Chapter 13: Laws of Static Fields

~Coulomb's Law and the Law of Universal Gravitation~

This chapter marks the beginning of our description of the theory of gravity. The fundamental method for constructing the theory of gravity is to describe it in a form similar to electrodynamics. Our approach is essentially analogous, and our model is the electrodynamics developed by Maxwell et al. Thanks to the existence of such a fine theory, our work is not so extensive, but to do so, we must inevitably study electrodynamics.

Coulomb's law in electrodynamics and the law of universal gravitation are very similar, and in fact, their mathematical formulations are almost identical. This has already been well studied, and it is something that 100 physicists would agree on. In this chapter, we will confirm this theory. In the next chapter, we will discover the concept of a gravitational magnetic field, but what is discussed in this chapter is necessary as a foundation.

13.1 Coulomb's Law

An object that carries an electric charge is called a charged object, and the amount of electric charge in such an object is called an electric charge. Generally, an electrically charged object has spatial extent, but an object that has no spatial extent at all, considered as a point, is called a point charge.

There are two types of electric charge: one is positive charge, and the other is negative charge.

In a vacuum, the force between two stationary charges is proportional to the product of their charges and inversely proportional to the square of the distance between them. The force repels the charges when the product is positive, and attracts them when it is negative. Assume that the charges are point charges. In equation form,

$$F \propto \frac{Q_1 Q_2}{r^2}$$

$$F = k\frac{Q_1 Q_2}{r^2} \qquad (13.1)$$

Here, F is the force between two charges, Q1,Q2 are the charges,

k is the proportionality constant, r is the distance between the charges.

If written in vector terms,

$$\mathbf{F}_2 = k\frac{Q_1 Q_2}{r^2}\mathbf{r}_{12} = k\frac{Q_1 Q_2}{r^2}\mathbf{r}_0 \qquad (13.2)$$

$$\mathbf{F}_1 = -\mathbf{F}_2 = -k\frac{Q_1 Q_2}{r^2}\mathbf{r}_{12} = k\frac{Q_2 Q_1}{r^2}\mathbf{r}_{21} = k\frac{Q_2 Q_1}{r^2}\mathbf{r}_0 \quad (\mathbf{r}_{12} = -\mathbf{r}_{21})$$

$\mathbf{F}_2$ is the force acting on Q_2, $\mathbf{F}_1$ is the force acting on Q_1

$\mathbf{r}_{12}$ is unit vector from Q_1 to Q_2

$\mathbf{r}_{21}$ is unit vector from Q_2 to Q_1

$\mathbf{r}_0$ is unit vector from the charge on the left side of the equation to the charge on the right side.

The direction of the force acting on Q and the direction of the force acting on Q_2 are always opposite, regardless of the positive and negative polarity combination of their respective charges. That is,

$$\mathbf{F}_1 = -\mathbf{F}_2 \qquad (13.3)$$

This relationship, as shown in equation (13.1), is called Coulomb's law.

The relationship between the positive and negative charges and the direction of force is,

Q_1	Q_2	$Q_1 Q_2$	$\mathbf{F}_1$	$\mathbf{F}_2$
+	+	+	←	→
−	−	+	←	→
−	+	−	→	←
+	−	−	→	←

Therefore, when the product of charges Q_1 and Q_2 is positive, it is a repulsive force, and when it is negative, it is an attractive force.

To quantitatively compare the quantity of electric charge, a unit quantity is necessary. However, electric charge is a relatively new quantity compared to mass, and before the discovery of Coulomb's law, there were no units for it. In such cases, there is arbitrariness between the proportionality constant k and the unit of electric charge; if the proportionality constant is arbitrarily determined, the unit quantity of electric charge is determined, and conversely, if the unit quantity of electric charge is arbitrarily determined, the proportionality constant is determined. There are infinitely many combinations of these, but currently, there are two types of units for electric charge that are commonly used: C (coulomb) and esu (ersted). The former is an MKS unit, and the latter is a CGS Gauss unit. The names are derived from the names of scientists. For coulomb, k in equation (13.1) is

$$k = \frac{1}{4\pi\varepsilon_0} = 8.98755179 \times 10^9$$

However, ε_0 is called the permittivity of vacuum, and it is a quantity of $8.85418782 \times 10^{-12}$.

This is the quantity when k is set as shown, and Ersted is the k in equation (13.1)

$$k = 1$$

This is the quantity when we set it as. It is not very meaningful to consider which of these two units is a more rational quantity. For example, there is no rational reason why the meter, the most common unit of length, is used, and there is no rational reason why the yard is used. In the end, unit quantities are just something that someone somewhere arbitrarily decided, and there is no rational reason for them. In the case of electric charge, it is perfectly fine if someone somewhere arbitrarily collects electric charge and claims that this amount is the unit quantity of electric charge.

In this case, there is basically no physical reason for setting $k = 8.98755179 \times 10^9$ or k = 1; they are simply chosen arbitrarily. However, it is true that the equation can be expressed more concisely depending on how the proportionality constant is set. For example, if $\varepsilon_0 = 1$, then k = 1/4 π , eliminating the need to define the quantity called electric flux density, which will be discussed later, and allowing the equation to be written more concisely.

Furthermore, while the unit of force, the Newton, is $1[\mathrm{N}] = 1[\mathrm{kgm/s^2}]$, it wouldn't be wrong to define it as, for example, $1[\mathrm{o}] = 0.12345[\mathrm{kgm/s^2}]$ and claim that [o] is a unit of force. However, since calculations would be cumbersome, no one would likely adopt such a unit.

The determination of units is usually determined historically. The law of universal gravitation is,

$$F = G\frac{Mm}{r^2}$$

It is written that a proportionality constant G = 6.67×10^{-11} is used, but if we set this to G = 1 and keep the units of force and distance unchanged, a new unit of mass can be defined. Using such a unit, the law of universal gravitation is,

$$F = \frac{Mm}{r^2}$$

It can be written as. No one would listen to an argument that the unit of mass should be changed because the equation is simpler when written this way. If such a change were allowed, all existing literature and industrial products that describe the law of universal gravitation and mass would have to be revised. No one wants to do such a troublesome thing, so such a change is avoided. Also, as is clear from this example, physical constants, like the way units are determined, have an element of arbitrariness.

If we believe that units in physics should be determined based on invariant physical phenomena rather than someone's arbitrary idea, then it is possible to incorporate many physical constants into the units. Even setting Coulomb's law as $\mathrm{k} = 8.98755179 \times 10^9$ is not particularly commendable, but if it is said that this is the historically accepted practice, then there is nothing to say in response.

Physically, it is sufficient to be able to compare the magnitudes of electric charges using appropriately defined units of charge. Using Coulomb's law, electric charge can be quantitatively determined whether it is measured in coulombs or ersteds. We will adopt the MKS unit system, which is a more general system of units. This system is convenient in terms of considering its correspondence with the law of universal gravitation. Coulomb's law using these units is:

$$\boldsymbol{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\boldsymbol{r}_0 \qquad (13.4)$$

This is the result.

13.2 Electric Field

Coulomb's law, like the law of universal gravitation, holds true under the approximation that the effect of the Coulomb force is transmitted instantaneously. This law generally does not hold true when electric charges are in motion. The laws described in this chapter are approximations where the electric charges are considered stationary. Such a static electric field is called an electrostatic field.

If we bring another charge q around a charge Q, a force will act on q according to Coulomb's law. Coulomb's law states that the magnitude of this force depends on the distance between charges Q and q, and even if the distance is the same, it will also depend on the amount of charge Q. This indicates that the effect of charge Q can be expressed as a spatial function. The way in which the Coulomb force exerts itself on its surroundings differs depending on the magnitude of the charge.

In other words, the characteristics of the electric field change. To express the electric field quantitatively, we need to consider the force that would act on a unit charge brought to a point in the electric field without disturbing the distribution of the electric field due to the original charge. Let's call this the strength of the electric field. To measure the strength of the electric field precisely, we need to bring in an extremely small amount of charge, mathematically an infinitesimal charge, and convert the force acting on it to a per-unit charge value. Just like in the gravitational field, the force acting on an infinitesimal charge is,

$$\mathbf{E} = \lim_{\Delta q \to 0} \frac{\mathbf{F}(\Delta \mathrm{q})}{\Delta \mathrm{q}} \tag{13.5}$$

This will be given as. If we convert this to per unit charge,

$$\mathbf{F} = \mathrm{q}\mathbf{E} \tag{13.6}$$

This means that $\mathbf{E}$ corresponds to equation (13.4),

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\boldsymbol{r}_0 \tag{13.7}$$

This E represents the strength of the electric field, or simply the electric field.

13.3 Electric Potential

Generally, the concept of work in physics is given in the form of the product of force and displacement, and if a force $\mathbf{F}$ acts from interval A to B, the work done W is

$$\mathrm{W} = -\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} \tag{13.8}$$

This is expressed as. The negative sign that appears here is a convenient one, determined by the arbitrarily chosen coordinate system. Substituting equation (13.6) into this equation,

$$W = -\int_A^B q\mathbf{E} \cdot d\mathbf{s} \qquad (13.9)$$

This allows us to define a quantity called work done on a Coulomb force field (electric field $\mathbf{E}$).

A special case of this equation is when the work required to transport a unit of electric charge is set to q = 1,

$$\varphi_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{s} \qquad (13.10)$$

It can be written as. This φ_{BA} is called the potential difference between point B and point A. Since the Coulomb force field is the same central force as the gravitational field, it is a conservative force field, and in a static electric field $\mathbf{E}$,

$$\mathrm{rot}\mathbf{E} = 0$$

This is true.

This quantity φ_{BA} is constant regardless of the path taken. That is, the integration path can be arbitrary, so if we assume that the integration path passes through point O, then equation (13.10) becomes:

$$\varphi_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{s} = -(\int_A^O \mathbf{E} \cdot d\mathbf{s} + \int_O^B \mathbf{E} \cdot d\mathbf{s})$$

It can be written as, and from the integral formula,

$$\int_A^O \mathbf{E} \cdot d\mathbf{s} = -\int_O^A \mathbf{E} \cdot d\mathbf{s}$$

and,

$$\varphi_{AO} = -\int_O^A \mathbf{E} \cdot d\mathbf{s}, \quad \varphi_{BO} = -\int_O^B \mathbf{E} \cdot d\mathbf{s}$$

If we set it as,

$$\varphi_{BA} = \varphi_{BO} - \varphi_{AO} \qquad (13.11)$$

This can be written as. This equation shows that if the potential difference with respect to any reference point O in the electric field is known, then the potential difference between any two points can be found by taking the difference in the potential differences of each point with respect to the reference point O. This reference point can be chosen at the most convenient point, and it is convenient to choose the point at infinity. The potential difference with respect to this point at infinity is,

$$\varphi = -\int_{\infty}^{r} \boldsymbol{E} \cdot d\boldsymbol{s} \qquad (13.12)$$

It can be written as, and this φ is simply called electric potential. This φ is the potential in electrodynamics.

13.4 Gradient of Electric Potential

Electric potential is a function of space, and as discussed in Chapter 10, we can consider the gradient of this quantity with respect to space. The relationship between this gradient of electric potential and the strength of the electric field is

$$\boldsymbol{E} = -\operatorname{grad}\varphi = -\nabla\varphi = -\left(\frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}\right) \qquad (13.13)$$

It is written as. Since the quantity obtained by spatially integrating $\mathbf{E}$ is φ, it is natural that the spatial derivative of φ is $\mathbf{E}$, and equation (13.13) is another way of expressing equation (13.12). Forms like equation (13.12) are called integral forms, and expressions like equation (13.13) are sometimes called differential forms.

13.5 Charge Distribution

Up until now, we have considered the distribution of electric charge as a point, and have not considered its extent. Here, let's generalize to a continuous charge distribution. Since it is clear that electric charge exists in a spatial location, in order to consider the distribution of electric charge, we must consider its volume distribution. This quantity is the amount of charge in a unit volume and is called the charge density ρ. This ρ is a spatial scalar function, and its unit is [charge/volume]. Expressing ρ as a function of x,y,z coordinates, we can write ρ(x,y,z). The product of this quantity and an infinitesimal volume dxdydz, ρ(x,y,z)dxdydz, is the amount of charge contained in that infinitesimal volume.

Let Q be the total charge contained within a domain V. If we write the minimum volume as dv=dxdydz, and integral ρ(x,y,z)dv over that domain, the resulting quantity is the total charge. That is,

$$Q = \int_v \rho dv \qquad (13.14)$$

13.6 Electric Flux and Gauss's Law

Let's define the quantity obtained by setting ε_0 in equation (13.7) to 1. This quantity is called electric flux density.

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{r}_0 \qquad (13.15)$$

As is clear from the definition, in a vacuum,

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \qquad (13.16)$$

Now, assuming that Q is inside a closed surface of any shape, such as a balloon, we consider the area integral of area vectors $d\mathbf{S}$ and $\mathbf{D}$, whose magnitude is equal to the area element, in a direction perpendicular to a very small number of surfaces on that closed surface (hereinafter referred to as area elements). The area integral over the entire closed surface is:

$$\Phi = \int_S \mathbf{D} \cdot d\mathbf{S} \qquad (13.17)$$

This can be written as. This Φ is called electric flux. In a simple case, if the closed surface is a sphere of radius r and Q is at its center, then $\mathbf{D}$ and $d\mathbf{S}$ are parallel,

$$\mathbf{D} \cdot d\mathbf{S} = DdS\cos 0 = DdS$$

Therefore, if we convert this to a unit area, the electric flux density D is $\frac{Q}{4\pi r^2}$ (it is called electric flux density because it is the amount of electric flux per unit area), and the surface area of the sphere is $4\pi r^2$, so the total electric flux is,

$$\emptyset = \frac{Q}{4\pi r^2} \times 4\pi r^2 = Q$$

Therefore, electric flux and electric charge are equal, and their quantities do not depend on the radius of the sphere.

This is the same for the total electric flux passing through an arbitrary closed surface as it is for a sphere. For example, this can be intuitively understood by imagining Q as a source from which particles, such as bullets, are emitted in all directions at a constant rate. Mathematically, it is not difficult, because the area of the orthogonal components of the area vectors $d\mathbf{S}$and $\mathbf{D}$ is given by the dot product (Figure 13.1).

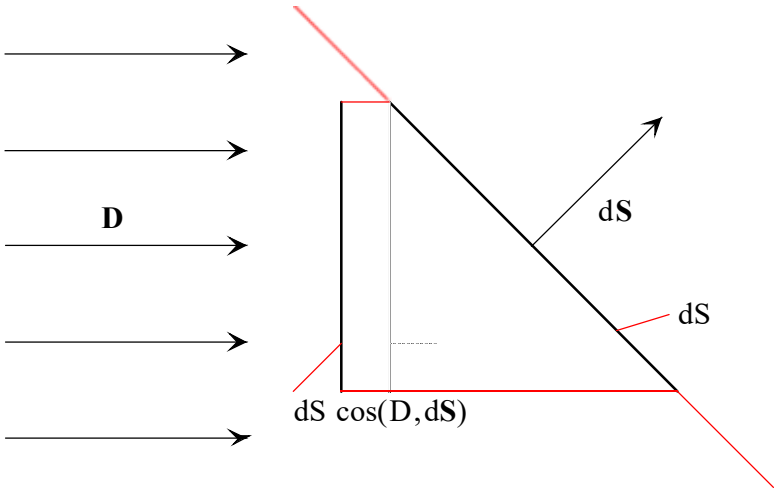

D
dS
dS
dS cos(D,dS)

Figure 13.1 Meaning of $\mathbf{D} \cdot d\mathbf{S}$

therefore,

$$\Phi = \int_S \mathbf{D} \cdot d\mathbf{S} = Q \qquad (13.18)$$

Also, from equation (13.14),

$$\Phi = \int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv \qquad (13.19)$$

This relationship can be obtained. Also, from equation(13.16),

$$\Phi = \int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho \, dv \qquad (13.20)$$

It can also be written as. This relationship is called Gauss's law. In words, the surface integral of the electric flux density $\mathbf{D}$ passing through a closed surface over the entire surface is equal to the total charge inside that closed surface. This law is an extension of Coulomb's law, and Gauss's law and Coulomb's law are not different physical laws, but rather different expressions of the same law. Using a collective expression, Coulomb's law is included in Gauss's law.

13.7 Differential Form of Gauss's Law

According to Gauss's divergence theorem, equation (11.27), for any vector **A**,

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_V \mathrm{div}\mathbf{A} dv \qquad (13.21)$$

There was a relationship between these two equations. From this equation and equation (13.20),

$$\int_V \mathrm{div}\boldsymbol{E} dv = \frac{1}{\varepsilon_0}\int_V \rho dv$$

It can be written as. Therefore,

$$\mathrm{div}\boldsymbol{E} = \nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad (13.22)$$

We obtain the following equation. This equation is the differential form of Gauss's law. This equation also holds true for a moving electric field. Gauss's law can be thought of as something like a bullet continuously emitted radially at a constant rate, but for example, the number of bullets fired in a given time does not differ between a stationary tank firing bullets from a machine gun and a moving tank firing bullets.

Substituting equation (13.13) into this equation,

$$\mathrm{div\ grad}\ \varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \qquad (13.23)$$

This is the result. This equation is called Poisson's equation.

13.8 Electrodynamic Expression of the Law of Universal Gravitation

The laws of static electric fields described so far can be applied almost directly to the laws of static gravitational fields. Let us list the laws of static gravitational fields (laws of universal gravitation) here, taking into account their correspondence with the laws of static electric fields.

(1) Relationship between the strength of the gravitational field and the gradient of the gravitational potential

$$\boldsymbol{E}_g = -\nabla\varphi_g \tag{13.24}$$

$\boldsymbol{E}_g$ is strength of the gravitational field, φ_g is gravitational potential

(2) Gauss's Law in a Gravitational Field

$$\nabla \cdot \mathbf{E}_g = \frac{\rho_g}{G_0} \tag{13.25}$$

ρ_g is the mass density, G_0 is an alternative form of the gravitational constant (= 1/4πG)

(3) Poisson's equation in a gravitational field

$$\nabla^2\varphi_g = -\frac{\rho_g}{G_0} \tag{13.26}$$

From these points, it can be seen that the laws of a static electric field and the laws of a static gravitational field are formally identical. However, it goes without saying that the crucial difference is that electricity is composed of two types of charges, positive and negative, while gravity is composed of one type of positive mass. Gauss's law in a gravitational field is another expression of the law of universal gravitation, and these equations constitute the entirety of the laws of a static gravitational field.