Chapter 11: Vector Analysis

This chapter will discuss the mathematics related to vector analysis.

This mathematics is not easy, but it is essential for field theories such as gravitational fields, electromagnetic fields, and fluid dynamics, so it cannot be avoided. The mathematics discussed in this chapter may be a headache for those who are not good at math, but I have tried to explain it as clearly as possible. What is written in this chapter is purely mathematical, so those who already have this knowledge do not need to read this chapter.

11.1 Scalar Gradient

In Chapter 10, we showed that the change in velocity of an object in space can be differentiated with respect to space. In general, we can consider the differentiation of any scalar function $\varphi(\mathrm{x},\mathrm{y},\mathrm{z})$ in space with respect to space. The differentiation of this function in the x, y, and z directions is called a partial derivative, and if we write $\varphi(\mathrm{x},\mathrm{y},\mathrm{z})$ as φ,

$$\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}$$

It is written as. If we write this quantity in vector form, and let that vector be A,

$$\mathbf{A}=\frac{\partial\varphi}{\partial \mathrm{x}}\boldsymbol{i}+\frac{\partial\varphi}{\partial \mathrm{y}}\boldsymbol{j}+\frac{\partial\varphi}{\partial \mathrm{z}}\boldsymbol{k}=\operatorname{grad}\varphi \qquad (11.1)$$

However, **i, j**, and **k** are unit vectors in the x, y, and z directions, respectively.

It is written as. $\mathrm{grad}\varphi$ is,

$$\mathrm{grad}\varphi=\nabla\varphi$$

It is sometimes written as ∇. ∇ is called the Hamiltonian operator, and this is a formal vector.

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

Considering this, this operator is often used because it simplifies the expression of mathematical formulas.

In general, a spatial function $\varphi(x,y,z) = 0$ becomes the equation of a surface, so this curve $\frac{\partial \varphi}{\partial x}$ is the inclination of the face in the x-direction.To calculate this, you basically need to set x, y, and z and specify the location in order to know the specific slope. However, if you set specific values for $y = y_1$ and $z = z_1$, excluding x...Then φ becomes (x,y_1,z_1) and can be considered as $\varphi(x)$ and differentiated with respect to x. After this, even if we set values such as $x = x_1$, we can find the slope in the x direction at that location. Since y_1 and z_1 are arbitrary constants, to partially differentiate $\varphi(x,y,z)$ with respect to x, we can simply treat y and z as constants and perform the calculation. The same applies to y and z.

For example, φ is

$$\varphi = xy^2 + 3zy + 2x^2z$$

Therefore, the partial derivatives in each direction are:

$$\frac{\partial \varphi}{\partial x} = y^2 + 4zx$$
$$\frac{\partial \varphi}{\partial y} = 2xy + 3z$$
$$\frac{\partial \varphi}{\partial z} = 3y + 2x^2$$

It can be calculated as. The slope in each direction at a specific location can be found by substituting values afterward. For example, at location (1, 2, 1),

$$\frac{\partial \varphi}{\partial x} = 8, \frac{\partial \varphi}{\partial y} = 7, \frac{\partial \varphi}{\partial z} = 8$$

Therefore, $\nabla\varphi$ in this case is

$$\nabla\varphi = 8\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$$

Therefore, we can determine the direction and magnitude of this vector in space.

11.2 Differentiation of Indirect Functions

In the function $u = f(x,y)$, assume that x and y are dependent on the independent variable t. That is, x and y are not independent variables, but $x = g(t), y = h(t)$ which functions of the independent variable t. At this time, $\frac{du}{dt}$ is,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad (11.2)$$

This can be written as. This theorem is well known as the chain rule for differentiation.

Next, let's consider the special case where the variable x takes on the role of the independent variable t. Assume that the function $u = f(x,y)$ is directly dependent on the independent variable x and indirectly dependent on the variable y, which is a function of x, and that there is a functional relation $y = g(x)$. If we write equation (11.2) with dx instead of dt,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad (11.3)$$

This holds true. Similarly, for a three-dimensional function $v = f(x,y,z)$, when $z = g(x,y)$, then, $\frac{\partial y}{\partial x}$ is,

$$\frac{\partial y}{\partial x} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial g(x,y)}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial g(x,y)}{\partial x} \qquad (11.4)$$

This holds true.

11.3 Multiple Integrals

Assume that a substance is distributed in a planar region S with a density of scalar point functions $\rho(x,y)$. If this region S is divided into n parts, the total area of region S is the sum of the infinitesimal regions $DxDy$, so the area of S is:

$$S = Dx_1Dy_1 + Dx_2Dy_2 + ... + Dx_nDy_n$$

It can be written as.

If the densities at each location within region S are $\rho(x_1,y_1), \rho(x_2,y_2)$, etc., and the total amount of substance is M, then density $\times$ area of the infinitesimal region = amount of substance in the infinitesimal region,

$$M \approx \rho(x_1,y_1)Dx_1Dy_1 + \rho(x_2,y_2)Dx_2Dy_2 + ... + \rho(x_n,y_n)Dx_nDy_n$$

This can be written as. By increasing the number of divisions and reducing the area of the infinitesimal region, M can be calculated accurately. That is,

$$M = \lim_{n\to\infty} [\rho(x_1,y_1)Dx_1Dy_1 + \rho(x_2,y_2)Dx_2Dy_2 + ... + \rho(x_n,y_n)Dx_nDy_n]$$

This type of calculation is called integration, and with respect to the region S,

$$M=\iint_S \rho(x,y)dxdy=\int_S \rho(x,y)dS \qquad (11.5)$$

This is written as. We have been considering DxDy as a single variable DS representing the area of an infinitesimal region, but if we consider Dx and Dy as separate variables, this integral becomes an integral with respect to two variables. An integral with respect to two or more variables in this way is called a multiple integral.

Next, let's consider how to calculate this multiple integral. If we plot the plane region S on the x,y coordinate system, we can write it as shown in Figure 11.1(1).

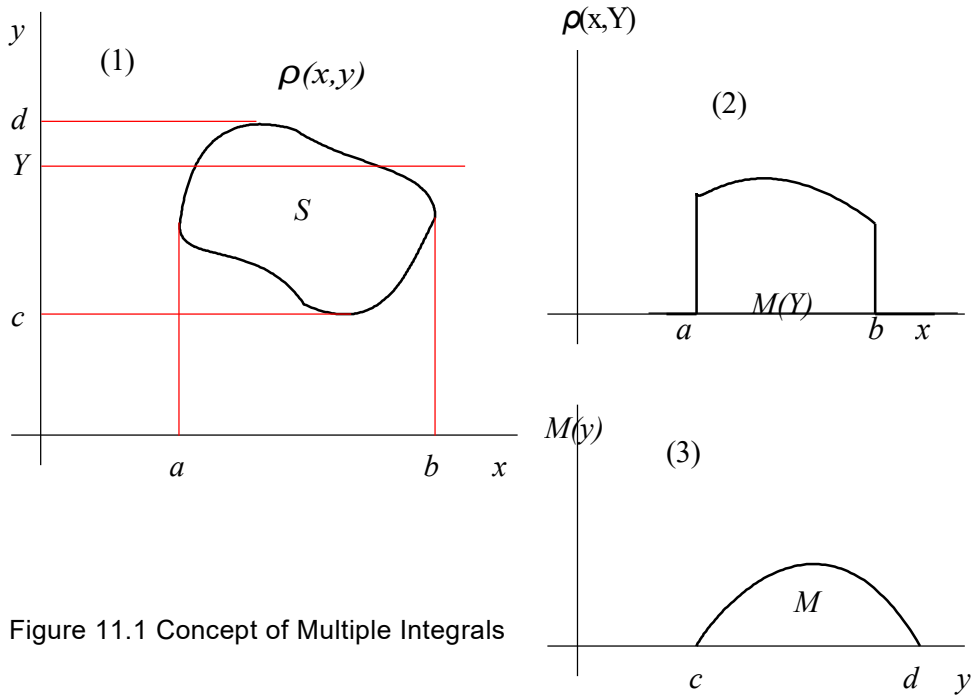


Figure 11.1 Concept of Multiple Integrals

The density of the region obtained by cutting out this region S with y=Y (constant) is ρ(x, Y). This density changes with the value of x, and its graph can be drawn as shown in Figure 11.1(2). Integrating this ρ(x, Y) with respect to x gives the amount M(Y) of substance y=Y. That is,

$$M(Y) = \int_a^b \rho(x, Y)dx \qquad (11.6)$$

Y is cSince it is any y from to d, if we write M(Y) as M(y) and draw its graph, we can write it as shown in Figure 11.1(3).

Integrating with respect to this, the resulting quantity is the total amount of substance M. In other words,

If we integrate this M(y) with respect to y, the resulting quantity is the total mass M.

$$M = \int_c^d M(y)dy \qquad (11.7)$$

Therefore, if we substitute equation (11.6) into equation (11.7),

$$M = \int_c^d \int_a^b \rho(x,y)dxdy$$

Therefore, this equation is equal to equation (11.5),

$$\iint_S \rho(x,y)dxdy = \int_c^d \int_a^b \rho(x,y)dxdy \qquad (11.8)$$

This can be written as, and the calculation of the double integral can be done by repeating the simple integral twice. In this case, when integrating $\rho(x,y)$ with respect to x, we can treat y as a constant, just as in the case of partial differentiation.

This can be extended to triple integration, which is the integration of a spatial domain. For example, suppose a substance is distributed in a three-dimensional domain v with density $\sigma(x,y,z)$. If we set v to z=Z (constant) and cut off v, the resulting cross-section is a planar domain S, the density of this planar domain is $\sigma(x,y,Z)$, and the density of the planar domain can be considered to be $\rho(x,y)$. From there, it is the same as in the case of double integration, so triple integration is,

$$\iiint_v \sigma(x,y,z)dxdydz = \int_e \int_c^d \int_a^b \sigma(x,y,z)dxdydz \qquad (11.9)$$

It can be calculated as. Furthermore, it is possible to consider $dxdydz$ as a single variable dv, which is the volume of a very small solid region, similar to how double integrals work.

$$\iiint_v \sigma(x,y,z)dxdydz = \int_v \sigma(x,y,z)dv \qquad (11.10)$$

It is sometimes written as such.

11.4 Line Integrals

In physics, when the direction of a force coincides with the direction of its displacement, the quantity [force of a constant magnitude] $\times$ [distance of displacement due to the force] is defined as work. That is,

$$W = F \times s \qquad (11.11)$$

W is work, F is force, s is distance

This quantity of work is quite different from the commonly used meaning of the word "work," and it may not be very appropriate to call such a quantity "work," but that's what it's called in physics. In physics, work is force $\times$ distance. Regardless of whether we call such a quantity "work" or something else, there is meaning in defining such a quantity.

Since the distance s can be divided into several smaller distances, Let Δs be the infinitesimal distance, and if we divide it into n intervals,

$$s = \Delta s_1 + \Delta s_2 + ... + \Delta s_n$$

This can be written as. Substituting this into equation(11.11),

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$$s = \Delta s_1 + \Delta s_2 + ... + \Delta s_n$$

This can be written as. Substituting this into equation (11.11),

$$W = F \times (Ds_1 + Ds_2 + ... + Ds_n)$$

$$= F \times Ds_1 + F \times Ds_2 + \ldots + F \times Ds_n$$

It can be written as. If we write the minute work $F \times Ds_1$ as DW_1,

$$W = DW_1 + DW_2 + ... + DW_n \qquad (11.12)$$

This can be written as, and the total work W can be expressed as the sum of infinitesimal work DW_1, DW_2, ...

How can we represent work in the general case where the magnitude and direction of the force are not constant, and the direction of the force does not coincide with the direction of displacement? The motion of an object that can be considered as a point generally traces a curved trajectory in space. The direction of the tangent at each point on this curve is equal to the direction of the object's displacement, and we will write the unit vector of this direction as $\mathbf{n}$, and the force will be written as a vector $\mathbf{F}$ as a variable. If we let $\mathbf{F}_p$ be the force at point P on this curve, and $\mathbf{n}_p$ be the unit vector in the tangent direction, then the dot product of $\mathbf{F}_p$ and $\mathbf{n}_p$ is the component of $\mathbf{F}_p$ in the direction of motion. That is,

$$\mathbf{F}_p \text{ 's movement direction component} = \mathbf{F}_p \cdot \mathbf{n}_p$$

This is the case (Figure 11.2).

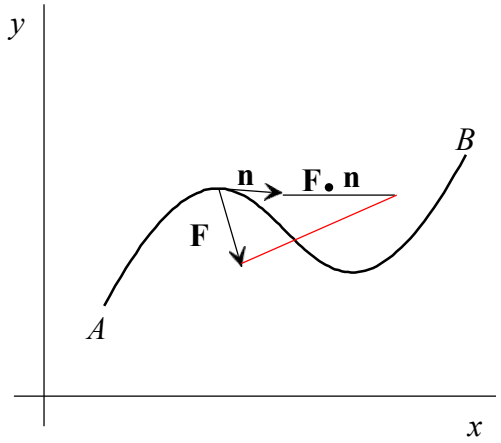


Figure 11.2 Tangential component of a vector

If we take a small length $\mathrm{D}s$ in the same direction as vector $\mathbf{n}$, then the component of $\mathbf{F}_{\mathrm{p}}$ in the direction of motion and $\mathrm{D}s$ lie collinear, and the product of the component of $\mathbf{F}_{\mathrm{p}}$ in the direction of motion and $\mathrm{D}s$ can be considered as if $\mathbf{F}_{\mathrm{p}}$ is locally constant, so we can use the relationship in equation (11.11). That is, the local work $\mathrm{D}W_{\mathrm{p}}$ at point P is,

$$\mathrm{D}W_{\mathrm{p}} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{n}_{\mathrm{p}}\mathrm{D}s_{\mathrm{p}} \qquad (11.13)$$

It can be written as.

Now, let's say we want to find the work W_{AB} done from point A to point B on the curve. If we divide this interval into m parts and write each part in the form of equation (11.13),

$$\mathrm{D}W_1 = \mathbf{F}_1 \cdot \mathbf{n}_1\mathrm{D}s_1$$

$$\mathrm{D}W_2 = \mathbf{F}_2 \cdot \mathbf{n}_2\mathrm{D}s_2$$

$$\vdots$$

$$\mathrm{D}W_{\mathrm{m}} = \mathbf{F}_{\mathrm{m}} \cdot \mathbf{n}_{\mathrm{m}}\mathrm{D}s_{\mathrm{m}}$$

Therefore, the work W_{AB} is approximately the sum of these,

$$\begin{aligned} W_{AB} &\approx DW_1 + DW_2 + ... + DW_m \\ &\approx \mathbf{F}_1 \,.\, \mathbf{n}_1 Ds_1 + \mathbf{F}_2 \,.\, \mathbf{n}_2 Ds_2 + ... + \mathbf{F}_m \,.\, \mathbf{n}_m Ds_m \end{aligned}$$

This can be written as. DW_1, DW_2, ... change in a stepwise manner, and to find W_{AB} precisely, we can take the limit of $m \to \infty$. That is,

$$\begin{aligned} W_{AB} &= \lim_{m\to\infty} (DW_1 + DW_2 + ... + DW_m) \\ &= \lim_{m\to\infty} (\mathbf{F}_1 \,.\, \mathbf{n}_1 Ds_1 + \mathbf{F}_2 \,.\, \mathbf{n}_2 Ds_2 + ... + \mathbf{F}_m \,.\, \mathbf{n}_m Ds_m) \end{aligned} \qquad (11.14)$$

This type of calculation is called a line integral.

$$W_{AB} = \int_A^B \mathbf{F} \cdot \mathbf{n} ds \qquad (11.15)$$

It is written as. This formula is also,

$$W_{AB} = \int_A^B \mathbf{F} \cdot d\mathbf{s} \qquad (11.16)$$

It is sometimes written as. Up until now, we have considered $\mathbf{F}$ as a vector representing force, but mathematically, an expression like equation (11.16) is possible for any vector.

If the path is a closed curve c,

$$W_c = \oint_c \boldsymbol{F} \cdot d\boldsymbol{s} \qquad (11.17)$$

It is sometimes written like this.

Furthermore, extremely small amounts of vector ds are

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

Since it can be written as, equation (11.16) is,

$$\begin{aligned} W_{AB} &= \int_A^B (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_A^B (F_x\,dx + F_y\,dy + F_z\,dz) \end{aligned} \qquad (11.18)$$

It can be written as. If $A(A_x, A_y, A_z)$ and $B(B_x, B_y, B_z)$, then equation (11.18) is,

$$W_{AB} = \int_{A_x}^{B_x} F_x\,dx + \int_{A_y}^{B_y} F_y\,dy + \int_{A_z}^{B_z} F_z\,dz \qquad (11.19)$$

This can be written as, and each term on the right-hand side is a regular definite integral, so the total work W_{AB} can be found by calculating the components in each direction individually. As is clear from this, the formula for definite integrals holds for line integrals.

Work and energy are often confused, but as pointed out in Chapter 8, work is always zero when an object does not move, but an object may not move even when a force is applied, and in such cases energy is still required. Therefore, just because the amount of work is zero does not mean that the energy applied is zero. Work can be positive or negative depending on the direction of the force, but the amount of work is

$$\mathbf{F} + (-\mathbf{F}) = 0$$

It is a quantity that holds true when it can be considered as such. Physical quantities include,

$$|\mathbf{F}| + |-\mathbf{F}| = 2F$$

We must not forget that such quantities also exist.

11.5 Area integral

Consider the area vector $\mathrm{D}S\mathbf{n} = \mathrm{D}\mathbf{S}$, which is the product of the area of a plane $\mathrm{D}S$ and the unit normal vector $\mathbf{n}$ perpendicular to this plane. Since there are three types of vector operations−scalar product, dot product, and cross product−we can consider the product of this vector with an arbitrary scalar function φ, and the dot product and cross product with an arbitrary function vector $\mathbf{A}$. That is,

$$\begin{array}{l}\varphi \mathrm{D}\mathbf{S} \\ \mathbf{A}\cdot \mathrm{D}\mathbf{S} \\ \mathbf{A}\times \mathrm{D}\mathbf{S}\end{array} \qquad (11.20)$$

We can consider this quantity.

Given a surface S, this surface can be divided into infinitesimal surfaces DS, and if the number of divisions is m, then the total area S is:

$$\mathrm{S} = \mathrm{DS}_1 + \mathrm{DS}_2 + ... + \mathrm{DS}_\mathrm{m}$$

This can be written as. For each term on the right-hand side, if the number of partitions is sufficiently large, then each infinitesimal surface can be considered as a plane, and we can consider the area vector $\mathrm{D}\mathbf{S}$, and its sum is,

$$\mathrm{DS}\mathbf{n}_1 + \mathrm{DS}\mathbf{n}_2 + ... + \mathrm{DS}\mathbf{n}_\mathrm{m} = \mathrm{D}\mathbf{S}_1 + \mathrm{D}\mathbf{S}_2 + ... + \mathrm{D}\mathbf{S}_\mathrm{m}$$

This sum is the sum of area vectors on infinitesimal surfaces of the surface S, but it cannot be considered as the area vector $\mathbf{S}$ of the total area S. This is because area vectors can only be defined on a plane. If we let $\mathbf{S}^{\prime}$ be the vector of the sum of area vectors on this infinitesimal surface, then approximately,

$$\mathbf{S}^{\prime} \approx D\mathbf{S}_1 + D\mathbf{S}_2 + ... + D\mathbf{S}_m$$

This can be written as. By increasing the number of partitions, the accuracy of the approximation can be increased indefinitely, and in the limit as $m \to \infty$, $\mathbf{S}^{\prime}$ can be precisely considered to be the sum of the individual area vectors of $\mathbf{S}$.

$$\mathbf{S}^{\prime} = \lim_{m \to \infty} (D\mathbf{S}_1 + D\mathbf{S}_2 + ... + D\mathbf{S}_m)$$

Therefore, if we substitute quantities like those in equation (11.20) into the right-hand side of this equation, and write the φ at each location as φ_1, φ_2, and the vector $\mathbf{A}$ as $\mathbf{A}_1, \mathbf{A}_2$, etc.,

$$\lim_{m \to \infty} (\varphi_1 D\mathbf{S}_1 + \varphi_2 D\mathbf{S}_2 + \ldots + \varphi_m D\mathbf{S}_m)$$

$$\lim_{m \to \infty} (\mathbf{A}_1 \cdot D\mathbf{S}_1 + \mathbf{A}_2 \cdot D\mathbf{S}_2 + ... + \mathbf{A}_m \cdot D\mathbf{S}_m)$$

$$\lim_{m \to \infty} (\mathbf{A}_1 \times D\mathbf{S}_1 + \mathbf{A}_2 \times D\mathbf{S}_2 + ... + \mathbf{A}_m \times D\mathbf{S}_m)$$

This type of calculation is called a surface integral, and each of them is:

$$\int_S \varphi d\mathbf{S} = \lim_{m\to\infty} (\varphi_1 D\mathbf{S}_1 + \varphi_2 D\mathbf{S}_2 + ... + \varphi_m D\mathbf{S}_m) \quad (11.21)$$

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \lim_{m\to\infty} (\mathbf{A}_1 \cdot D\mathbf{S}_1 + \mathbf{A}_2 \cdot D\mathbf{S}_2 + ... + \mathbf{A}_m \cdot D\mathbf{S}_m) \quad (11.22)$$

$$\int_S \mathbf{A} \times d\mathbf{S} = \lim_{m\to\infty} (\mathbf{A}_1 \times D\mathbf{S}_1 + \mathbf{A}_2 \times D\mathbf{S}_2 + ... + \mathbf{A}_m \times D\mathbf{S}_m) \quad (11.23)$$

It is written as. Normally, a surface integral refers to equation (11.22), but equations (11.21) and (11.23) are also types of surface integrals.

The surface integrals on a closed surface are,

$$\oint_s \varphi d\mathbf{S}$$

$$\oint_s \mathbf{A} \cdot d\mathbf{S}$$

$$\oint_s \mathbf{A} \times d\mathbf{S}$$

It is sometimes written that the direction of the unit normal vector is defined as outward.

The unit normal vector is given by, as stated in equation (6.8),

$$\mathbf{n} = \mathbf{i}\cos(n,x) + \mathbf{j}\cos(n,y) + \mathbf{k}\cos(n,z)$$

It can be written as, and since $d\mathbf{S} = \mathbf{n}dS$, equation (11.22) is,

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_S (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (\mathbf{i}\cos(n,x) + \mathbf{j}\cos(n,y) + \mathbf{k}\cos(n,z))dS$$

$$= \int_S (A_x\cos(n,x) + A_y\cos(n,y) + A_z\cos(n,z))dS \quad (11.24)$$

It can also be written as. The area $dxdy$ of the projection of the area dS of a minimal plane onto the x,y plane changes depending on the angles n and z that it makes with the plane.(Figure 11.3)

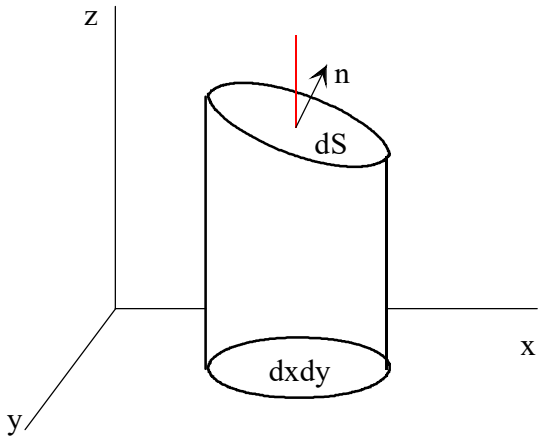


Figure 11.3 Projection of plane dS onto the x,y plane

Between them,

$$dS\cos(n,z) = dxdy$$

There is a relationship as, Similarly, for the dydz, dzdx plane,

$$dS\cos(n,x) = dydz$$

$$dS\cos(n,y) = dzdx$$

Because of this relationship, if we substitute this into equation (11.24),

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_S (A_x \frac{dydz}{dS} + A_y \frac{dzdx}{dS} + A_z \frac{dxdy}{dS}) dS$$
$$= \int \int_S (A_x dydz + A_y dzdx + A_z dxdy) \qquad (11.25)$$

It can also be written as

The physical meaning of equation (11.25) is the total amount of vector $\mathbf{A}$ passing through the surface S.

11.6 Gauss's Divergence Theorem

Let v be the region enclosed by a closed surface S. There exists a continuous function $r(x,y,z)$ in this region v, and this function has $\frac{\partial r(x,y,z)}{\partial z}$. For simplicity, we assume that the closed surface S intersects with the line parallel to the z-axis at only two points. Given x and y, the intersection points with the closed surface S are z_1 and z_2, and the inequality relationship is $z_1 < z_2$, the triple integral of this function $\frac{\partial r(x,y,z)}{\partial z}$ over the region v is:

$$\int_v \frac{\partial r(x,y,z)}{\partial z} dv = \iiint_{z1}^{z2} \frac{\partial r(x,y,z)}{\partial z} dzdxdy$$

$$= \iint_{S_{xy}} [r(x,y,z_2) - r(x,y,z_1)]dxdy$$

$$= \iint_{S_{xy}} r(x,y,z_2)dxdy - \iint_{S_{xy}} r(x,y,z_1)dxdy$$

This is the case (Figure 11.4).

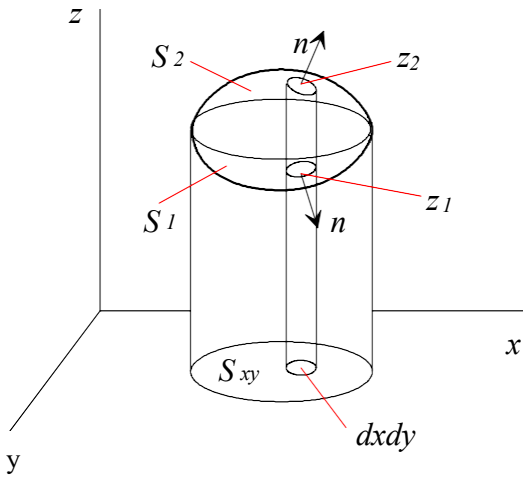


Figure 11.4 Gauss's divergence theorem

If dS is the area of a minimal surface on surface S, and dxdy is the area of its projection onto the x,y plane, then using the angle between the normal vector n on surface S and the z-axis,

At the intersection of surface S and z_2; $dS\cos(n,z) = dxdy$

At the intersection of surface S and z_1; $-dS\cos(n,z) = dxdy$

Because of the relationship,

$$\int_v \frac{\partial r(x,y,z)}{\partial z} dv = \int_{S_2} r(x,y,z_2)\cos(n,z)dS + \int_{S_1} r(x,y,z_1)\cos(n,z)dS$$

S_2 is the surface where the curved surface S and z2intersect.

S_1 is the surface where the curved surface S and z_1 intersect.

It can be written as. z_1 and z_2 are the range of z, and in this equation, the integral region is explicitly shown, so z_1 and z_2 can be written as z,

$$\int_v \frac{\partial r(x,y,z)}{\partial z} dv = \int_{S_2} r(x,y,z)\cos(n,z)dS + \int_{S_1} r(x,y,z)\cos(n,z)dS$$

Therefore, the right-hand side of this equation is the sum of the integrals over surface S_1 and S_2, and thus represents the integral over all surfaces S.

$$\int_v \frac{\partial r(x,y,z)}{\partial z} dv = \int_S r(x,y,z)\cos(n,z)dS$$

This is the result.

Similarly, if we consider other functions p(x,y,z) and q(x,y,z),

$$\int_v \frac{\partial p(x,y,z)}{\partial x} dv = \int_S p(x,y,z)\cos(n,x)dS$$
$$\int_v \frac{\partial q(x,y,z)}{\partial y} dv = \int_S q(x,y,z)\cos(n,y)dS$$

We get this. Adding these together,

$$\int_v \left(\frac{\partial p}{ax} + \frac{\partial q}{ay} + \frac{\partial r}{az}\right)dv = \int_S [p\cos(n,x) + q\cos(n,y) + r\cos(n,z)]dS$$

This is the result. This formula is known as Gauss's divergence theorem.

If we write this equation in the form of equation (11.25),

$$\int_v \left(\frac{\partial p}{ax} + \frac{\partial q}{ay} + \frac{\partial r}{az}\right)dv = \iint_S (pdydz + qdzdx + rdxdy)$$

It can also be written as. If p, q, and r are equal to the components of the vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, and $p = A_x$, $q = A_y$, and $r = A_z$,

$$\int_c \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)dv = \iint_S (A_x dydz + A_y dzdx + A_z dxdy) \quad (11.26)$$

It can be written as. The right-hand side of this equation is equal to equation (11.25),

$$\int_v \mathrm{div}\mathbf{A}dv = \int_S \mathbf{A} \cdot d\mathbf{S}$$

$$\mathrm{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (11.27)$$

This can be written as. This equation is the vector form representation of Gauss's divergence theorem.

11.7 Vector Divergence

In general, when the vector function $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$,

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This is called the divergence of $\mathbf{A}$, and is denoted by $\mathrm{div}\mathbf{A}$. This is a scalar. If we write $\mathrm{div}\mathbf{A}$ using the Hamiltonian operator ∇,

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot \left(A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}\right)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (11.28)$$

This is the result. If we write equation (11.27) in a different form,

$$\mathrm{div}\mathbf{A} = \lim_{\Delta v \to 0} \frac{\int_s \boldsymbol{A} \cdot d\boldsymbol{S}}{\Delta v}$$

This equation is the definition of divergence.

In a vector field, points where the divergence is not zero are called emergent points. Let S be a closed surface containing the emergent points of a vector field $\mathbf{A}$, and the volume integral with respect to the region v enclosed by this surface is calculated.

$$\int_v \mathrm{div}\mathbf{A}\mathrm{dv}$$

This is called the strength of the eruption point. According to Gauss's divergence theorem, this is the surface integral on S.

$$\int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

It is equal to.

This surface integral represents the number of streamlines extending outward from the curved surface S, and the number of these streamlines is proportional to the strength of the wellspring. divA is the strength of the wellspring per unit volume; the larger divA, the greater the number of streamlines extending outward from the surrounding curved surface S, and if divA=0, the number of streamlines extending outward is zero. For example, if a fluid such as water is welling up, and its velocity vector is $\mathbf{v}$, then $\mathrm{div}\,\mathbf{v} \neq 0$, and if there is no welling up, then divA=0. Generally, where there is no flow generation or extinction, the divergence is zero.

11.8 Relationship between the normal vector and direction cosine of a curved surface

When the equation of a surface S is z = f(x,y), according to the knowledge of analytical geometry, the curve The relationship between the normal vector n at a point (x,y,z) on plane S and the direction cosines of the x,y, and z axes is:

$$\cos(n,x) = \frac{\frac{\partial f}{\partial x}}{\pm\sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}}$$

$$\cos(n,y) = \frac{\frac{\partial f}{\partial y}}{\pm\sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}}$$

$$\cos(n,z) = \frac{-1}{\pm\sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}} \qquad (11.30)$$

It can be expressed as. From these equations,

$$\frac{\partial f}{\partial x}\ \cos(\mathrm{n},\mathrm{z}) = -\cos(\mathrm{n},\mathrm{x}) \qquad (11.31)$$

$$\frac{\partial f}{\partial y}\ \cos(\mathrm{n},\mathrm{z}) = -\cos(\mathrm{n},\mathrm{y})$$

It can be written as.

11.9 Green's Theorem in the Plane

Suppose there exists a continuous function $p(x,y)$ in a plane region S of a closed curve c, and this function has a continuous derivative $\frac{\partial p(x,y)}{\partial y}$. Assuming that the closed curve C in region S intersects with a line parallel to the y-axis at only two points, if we perform a double integral with respect to y over region S, then

$$\int_S \frac{\partial p(x,y)}{\partial y} dS = \int\int_S \frac{\partial p}{\partial y} dxdy = \int_a^b \int_{y_1}^{y_2} \frac{\partial p}{\partial y} dydx$$

$$= \int_a^b [p(x,y_2) - p(x,y_1)]dx = \int_a^b p(x,y_2)dx - \int_a^b p(x,y_1)dx$$

This is the result.(figure 11.5)

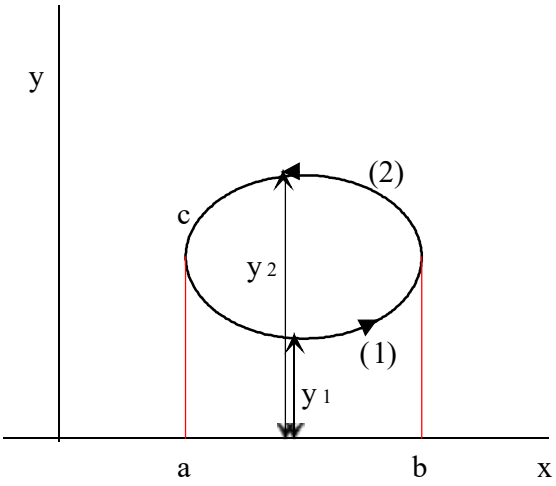


Figure 11.5 Green's Theorem

The right-hand side of this equation is the line integral x = a from a to b.

$$\int_a^b p(x,y_2)dx = -\int_b^a p(x,y_2)dx = -\int_{(2)} p(x,y)dx$$

$$-\int_a^b p(x,y_1)dx = -\int_{(1)} p(x,y)dx$$

Therefore,

$$\int_s \frac{\partial p(x,y)}{\partial y} dS = -\int_{(2)} p(x,y)dx - \int_{(1)} p(x,y)dx = -\int_c p(x,y)dx \quad (11.32)$$

Similarly, if we calculate for another function q(x,y),

$$\int_s \frac{\partial q(x,y)}{\partial x} dS = \int_c q(x,y)dy \qquad (11.33)$$

This is obtained. The reason why the signs of equations (11.32) and (11.33) are opposite is that if we take the direction of the line integral counterclockwise as shown in the figure, with respect to the x-axis, in path B where y is smaller, the direction of increase of x and the direction of the line integral are the same, whereas with respect to the y -axis they are opposite. Subtracting equation (11.32) from equation (11.33) gives,

$$\int_S (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dS = \int_c (pdx + qdy) \qquad (11.34)$$

This equation is Green's theorem in the plane.

11.10 Stokes' Theorem

Let's extend Green's theorem to any open surface S with a closed curve c. Assume that a line parallel to the z-axis intersects the surface S at only one point, and let $p(x,y,z)$ be a continuous function defined in the neighborhood of the surface S, with a continuous derivative $\frac{\partial p(x,y,z)}{\partial y}$. Then, the line integral of $p(x,y,z)$ is...

$$\int_c p(x,y,z)dx$$

The curve c lies on the surface S, and if the equation of this surface is $z=f(x,y)$, then the function p can be written as $p(x,y,f(x,y))$,

$$\int_c p(x,y,z)dx = \int_l p(x,y,f(x,y))dx$$

Here, l is the projection of the curve c onto the xy-plane. Since the right-hand side of this equation is a planar region, applying Green's theorem, from equation (11.32),

$$\int_c p(x,y,z)dx = -\int_{S_{xy}} \frac{\partial p(x,y,z)}{\partial y} dS_{xy} \qquad (11.35)$$

This is the case (Figure 11.6).

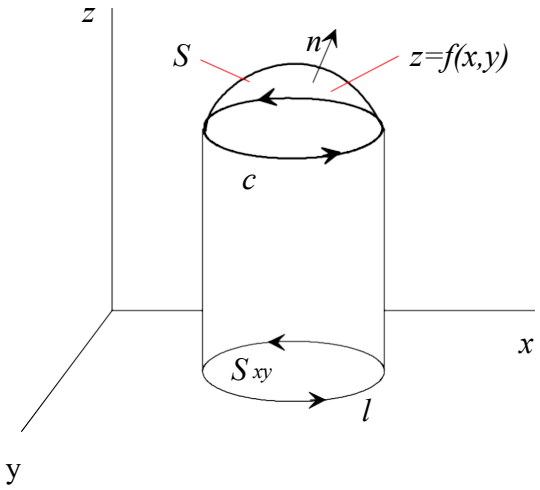


Figure 11.6 Stokes' Theorem

Here, S_{xy} is the projection of the surface S onto the xy-plane. The calculation of $\frac{ap(x,y,z)}{ay}$ is done using the differential formula for indirect functions (11.4):

$$\frac{\partial p(x,y,z)}{\partial y} = \frac{\partial p(x,y,z)}{\partial y} + \frac{\partial p(x,y,z)}{\partial z}\frac{\partial p(x,y)}{\partial y}$$

Therefore, equation (11.35) is,

$$\int_c p(x,y,z)dx = -\int_{S_{xy}} \left[\frac{\partial p(x,y,z)}{\partial y} + \frac{\partial p(x,y,z)}{\partial z}\frac{\partial p(x,y)}{\partial y}\right]dS_{xy} \qquad (11.36)$$

Between dS and dS_{xy} , the angle between the surface normal n and the z-axis is used.

$$dS_{xy} = \cos(n,z)dS$$

Because of this relationship, the right-hand side of equation (11.36) is $\frac{\partial p}{\partial y}$

$$-\int_{S_{xy}} \left[\frac{\partial p(x,y,z)}{\partial y} + \frac{\partial p(x,y,z)}{\partial z}\frac{\partial p(x,y)}{\partial y}\right]dS_{xy}$$
$$= \int_S \left[\frac{\partial p}{\partial y}\cos(n,z) - \frac{\partial p}{\partial z}\frac{\partial f(x,y)}{\partial y}\cos(n,z)\right]dS$$

This is the result. Furthermore, this term is derived from equation (11.31),

$$= \int_S \left[\frac{\partial p}{\partial z}\cos(n,y) - \frac{\partial p}{\partial y}\cos(n,z)\right]dS$$

Therefore,

$$\int_c p(x,y,z)dx = \int_S \left[\frac{\partial p}{\partial z}\cos(n,y) - \frac{\partial p}{\partial y}\cos(n,z)\right]dS \qquad (11.37)$$

To obtain. Similarly, for another function q(x,y, z) and r(x,y, z),

$$\int_c q(x,y,z)dy = \int_S \left[\frac{\partial q}{\partial x}\cos(n,z) - \frac{\partial q}{\partial z}\cos(n,x)\right]dS$$
$$\int_c r(x,y,z)dz = \int_S \left[\frac{\partial r}{\partial y}\cos(n,x) - \frac{\partial r}{\partial x}\cos(n,y)\right]dS$$

You can obtain this.

Adding the three resulting equations,

$$\int_c (pdx + qdy + rdz) = \int_S [(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}) \cos(n,x) + (\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}) \cos(n,y) + (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}) \cos(n,z)]dS \quad (11.38)$$

This formula is called Stokes' theorem.

If p, q, and r are equal to the components of vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, and $p = A_x$, $q = A_y$, $r = A_z$, then equation (11.38) becomes:

$$\int_c (A_x dx + A_y dy + A_z dz) = \int_S [(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) \cos(n,x) + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \cos(n,y) + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \cos(n,z)]dS \quad (11.39)$$

It can be written as. The left side of this equation is in the form of equation (11.18), so it can be expressed as a line integral of a vector.

$$\int_c (A_x dx + A_y dy + A_z dz) = \int_c \mathbf{A} \cdot d\mathbf{s}$$

It can be written as, and the right-hand side of equation (11.39) is in the form of equation (11.24), so it is a surface integral of a vector,

The right-hand side of equation (11.39) is $\int_S \mathrm{rot}\mathbf{A} \cdot d\mathbf{S}$

$$\mathrm{rot}\mathbf{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\mathbf{i} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\mathbf{j} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\mathbf{k}$$

1

It can be written as. Therefore, the vector representation of Stokes' theorem is,

$$\int_c \mathbf{A} \cdot d\mathbf{s} = \int_S \mathrm{rot}\mathbf{A} \cdot d\mathbf{S} \quad (11.40)$$

This is the result.

11.11 Rotation of a Vector

In general, the vector function $\mathbf{A}$ let $\mathbf{A} = \mathrm{A_x}\mathbf{i} + \mathrm{A_y}\mathbf{j} + \mathrm{A_z}\mathbf{k}$,

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial y} - \frac{\partial A_x}{\partial y}$$

A vector whose components are is called the rotation of **A** and is denoted by rot**A**. This is a vector. That is, if we express rot**A** using x,y,z coordinates,

$$\mathrm{rot}\mathbf{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\mathbf{i} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\mathbf{j} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\mathbf{k} \qquad (11.41)$$

$\mathrm{rot}\mathbf{A}$ is sometimes written as curl$\mathbf{A}$. Also, if written using the Hamiltonian operator,

$$\mathrm{rot}\mathbf{A} = (\frac{\partial}{\partial \mathrm{x}}\boldsymbol{i} + \frac{\partial}{\partial \mathrm{y}}\boldsymbol{j} + \frac{\partial}{\partial \mathrm{z}}\boldsymbol{k}) \times (\mathrm{A_x}\mathbf{i} + \mathrm{A_y}\mathbf{j} + \mathrm{A_z}\mathbf{k}) = \nabla \times \mathbf{A} \qquad (11.42)$$

It is also written as. If we write equation (11.40) in a different form,

$$\mathrm{rot}\mathbf{A} = \lim_{\Delta S \to 0} \frac{\int_c \boldsymbol{A} \cdot d\boldsymbol{s}}{\Delta S} \qquad (11.43)$$

This equation is the definition of rotation.

Where the rotation in a vector field is not zero, there is a vortex. In a vector field **A**, the divergence of rot**A**, div rot**A**, is

$$\begin{aligned}\text{div rot}\boldsymbol{A} &= \frac{\partial}{\partial x}(\text{rot}\boldsymbol{A})_x + \frac{\partial}{\partial y}(\text{rot}\boldsymbol{A})_y + \frac{\partial}{\partial z}(\text{rot}\boldsymbol{A})_z \\ &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} \\ &= 0\end{aligned}$$

Therefore, since it is always zero, the streamlines of rot**A** have no fountains, forming a closed loop and becoming a ring.

Where there is a vortex, the surface integral is on the surface S surrounded by the closed curve c.

$$\int_S \text{rot}\mathbf{A} \cdot d\mathbf{S}$$

This is called the strength of the vortex enclosed by the curve c, and according to Stokes' theorem, the line integral with respect to c is

$$\int_c \mathbf{A} \cdot d\mathbf{s}$$

It is equal to . As is clear from equation (11.43), for rot**A** = 0, this line integral must be zero. Conversely, for rot**A** + 0, this line integral must not be zero. For the integral around one turn not to be zero, the vector field must have a vortex.

11.12 Vector Formula

When k is a constant, C is a constant vector, ϕ and ρ are scalar functions, and **A**, **B**, and **D** are vector functions, the following formula holds.

$$\text{grad}\, k = 0 \tag{11.44}$$

$$\text{grad}\, k\varphi = k\, \text{grad}\, \varphi \tag{11.45}$$

$$\text{grad}(\varphi + \rho) = \text{grad}\, \varphi + \text{grad}\, \rho \tag{11.46}$$

$$\text{grad}\, \varphi\rho = \nabla(\varphi\rho) = \rho\nabla\varphi + \varphi\nabla\rho \tag{11.47}$$

$$\text{div}\, \mathbf{C} = 0 \tag{11.48}$$

$$\text{div}\, k\mathbf{A} = k\, \text{div}\mathbf{A} \tag{11.49}$$

$$\text{div}(\mathbf{A} + \mathbf{B}) = \text{div}\mathbf{A} + \text{div}\, \mathbf{B} \tag{11.50}$$

$$\text{div}\, \varphi\mathbf{A} = \nabla \cdot (\varphi\mathbf{A}) = \nabla\varphi \cdot \mathbf{A} + \varphi\nabla \cdot \mathbf{A} \tag{11.51}$$

$$\text{rot}\mathbf{C} = 0 \tag{11.52}$$

$$\text{rot}\, k\mathbf{A} = k\, \text{rot}\mathbf{A} \tag{11.53}$$

$$\text{rot}(\mathbf{A} + \mathbf{B}) = \text{rot}\mathbf{A} + \text{rot}\mathbf{B} \tag{11.54}$$

$$\text{rot}\varphi\mathbf{A} = \nabla \times (\varphi\mathbf{A}) = \nabla\varphi \times \mathbf{A} + \varphi\nabla \times \mathbf{A} \tag{11.55}$$

$$\text{rot}\ \text{grad}\varphi = \nabla \times \nabla\varphi = 0 \tag{11.56}$$

$$\text{div}\ \text{rot}\mathbf{A} = 0 \tag{11.57}$$

$$\text{rot}\ \text{rot}\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{11.58}$$

Furthermore, the following theorem also holds true.

When $\text{rot}\mathbf{A} = 0$

There exists a φ **such that** $\mathbf{A} = \nabla\varphi$ (11.59)

When $\text{div}\mathbf{B}=0$

There exists a vector $\mathbf{D}$ **such that** $\mathbf{B} = \text{rot}\mathbf{D}$ (11.60)