Chapter 10: Expression of Force

~Representing force using time and spatial derivatives of acceleration~

While acceleration is generally considered to be the time derivative of velocity, is there any other way to define acceleration? Besides the change in velocity with respect to time, there must also be a change in velocity with respect to space.

In this chapter, we will discover an alternative form of expression law equivalent to Newton's second law of motion. This law is superior to Newton's second law of motion in that it allows us to define acceleration as a function of position in space. Furthermore, this law will answer questions such as why we can consider acceleration that depends on position in space, such as gravitational acceleration, to be equal to the acceleration that is a derivative of acceleration with respect to time.

10.1 Measurement of Acceleration by Time Differential

Generally, acceleration is expressed as the time derivative of velocity. To distinguish this defined acceleration from other accelerations discovered later, we will call it time acceleration. The geometric representation of the relationship between velocity and time in this method can be expressed by drawing two orthogonal axes, with one axis representing velocity and the other representing time, assuming that the change in velocity is one-dimensional motion along a straight line. A common measurement method involves measuring the velocity of an object at appropriate time intervals, summarizing the results in a table, and plotting the points on a graph. If the measurement time interval is shortened, the points will almost connect and be represented as a curve (Figure 10.1).

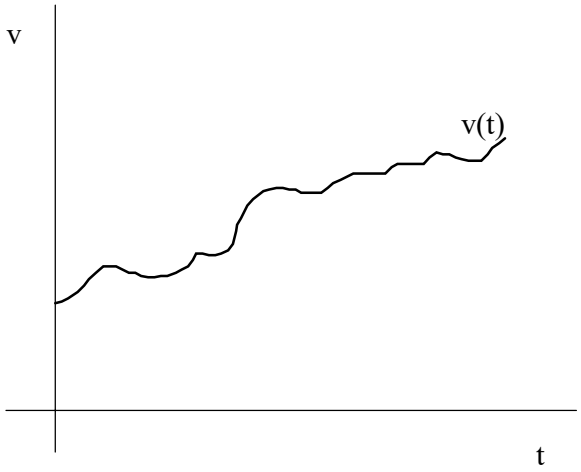


Figure 10.1 Relationship between speed and time

Time acceleration is, for example, the slope of each point (corresponding to that time) on the curve in Figure 10.1. This slope is equivalent to the incline of a mountain; a steeper slope indicates a larger time acceleration, while a gentler slope indicates a smaller time acceleration.

In other words, if we write the curve shown in Figure 10.1 as v(t), then its time acceleration **a** is:

$$\mathbf{a} = \frac{dv(t)}{dt} \qquad (10.1)$$

It is given by . This quantity of time acceleration can be expressed as a straight line for the motion of any object at a given time. No matter how complex the motion of an object may be, instantaneously, the direction of motion of the object can be considered to be a straight line, and that motion will either accelerate, decelerate, or remain unchanged, and this change can be expressed as the amount of the slope of the time acceleration. That is, the time acceleration at a given time can be expressed as a linear function of a minimum time change dt and a minimum velocity change dv taken in the direction of motion (Figure 10.2).

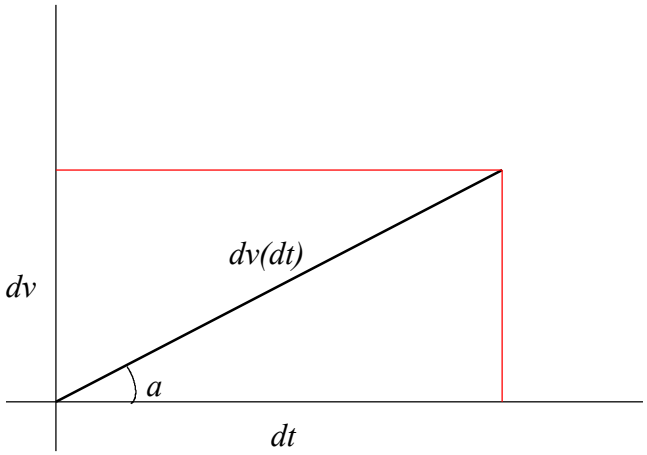


Figure 10.2 Relationship between minimal velocity change and minimal time change and time acceleration

10.2 Measurement of Acceleration by Spatial Differential

The method for measuring acceleration is not limited to the method described in 10.1. In 10.1, we considered the change in velocity with respect to time, but we can also consider the change in velocity with respect to space. For example, the velocity of a car moving in a straight line can be measured at a distance of 10m and at a distance of 20m, allowing us to consider the change in velocity with respect to space (distance).

Such changes can be represented graphically, for example, as shown in Figure 10.3.

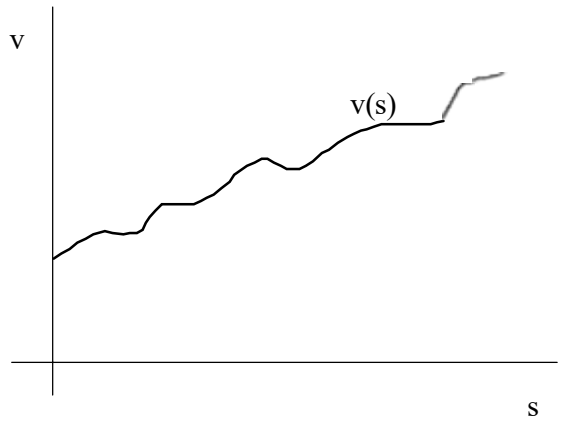


Figure 10.3 Relationship between speed and distance

10.3 Relationship between Acceleration by Time Differential and Acceleration by Spatial Differential

What is the relationship between the same motion when represented in the format of Figure 10.1 and when represented in the format of Figure 10.3? Let's try drawing a figure of uniformly accelerated motion with specific numerical values. In the right figure of Figure 10.4, after drawing the graph in the left figure, the relationship between v and s can be geometrically obtained by measuring the area of the region enclosed by the graph of v = v(t) and the v axis. Of course, it can also be obtained by directly measuring the velocity and distance of the moving object.

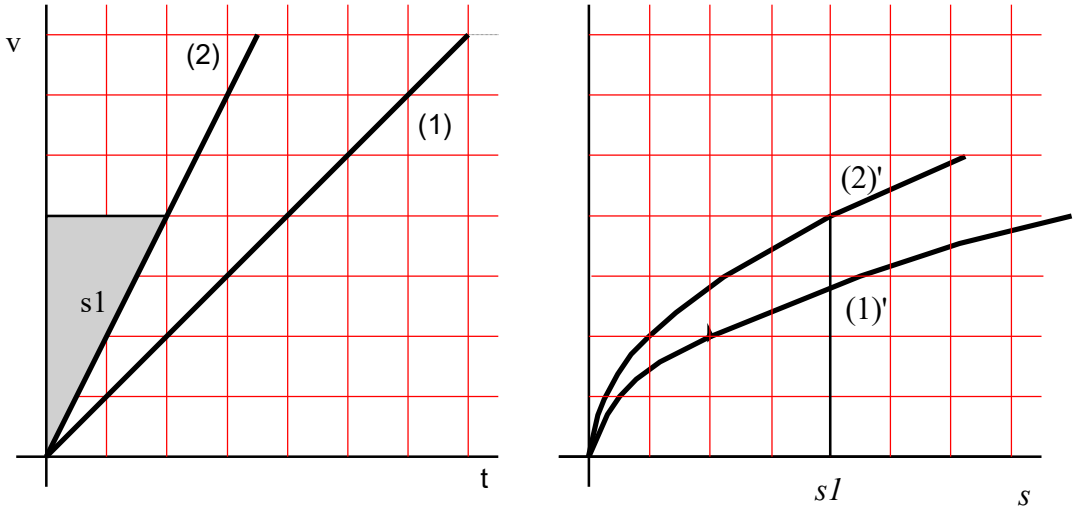


Figure 10.4 Graphs showing the same movement measured from different angles.

(1) $v = t$ (1)' $s = \frac{1}{2}v^2$

(2) $v = 2t$ (2)' $s = \frac{1}{4}v^2$

When moving with equal acceleration a, if the initial velocity is zero, generally,

$$v = at \tag{10.2}$$

This can be written as. a=1 for (1) and a=2 for (2), Using this a, the relationship between distance and velocity can be expressed as follows:

$$s = \frac{1}{2a}v^2 \tag{10.3}$$

This can be written as, and if we rewrite this equation,

$$\frac{1}{2}v^2 = as \tag{10.4}$$

This can be written as. If we graph equations (10.2) and (10.4), we get Figure 10.5.

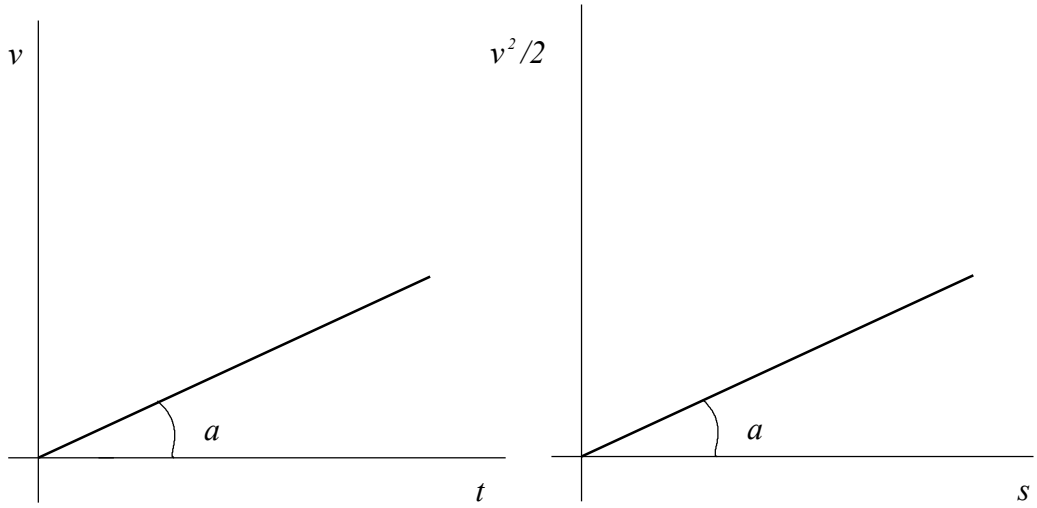


Figure 10.5 Two representations of uniformly accelerated motion

If we remove a from equations (10.2) and (10.4)

$$\frac{\frac{1}{2}v^2}{S} = \frac{v}{t} \qquad (10.5)$$

This relationship holds true in the case of uniformly accelerated motion.

On the other hand, the minimum time acceleration of a general motion with changing acceleration can be represented in Figure 10.2, and it can be assumed that the change in acceleration is constant over a minimum time. Similarly, the minimum distance $\mathrm{d}s$ and the minimum velocity change $\mathrm{d}(\frac{1}{2}\mathrm{v}^2)$ between them can be considered proportional, so the relationship between $\mathrm{d}(\frac{1}{2}\mathrm{v}^2)$ and $\mathrm{d}s$ can be plotted on a graph.Therefore, it can be written as shown in Figure 10.6.

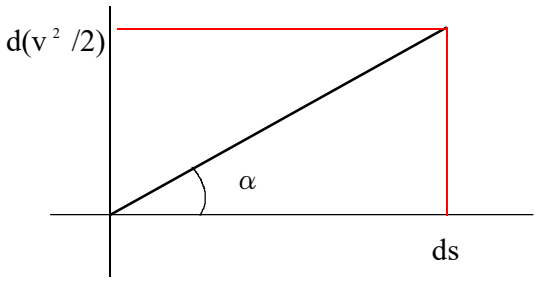


Figure 10.6 Graph of minimum acceleration in space

Therefore,

$$a = \frac{d(\frac{1}{2}v^2)}{ds} \qquad (10.6)$$

From the correspondence between Figures 10.2, 10.5, and 10.6, we can conclude that a = b.

$$\frac{d(\frac{1}{2}v^2)}{ds} = \frac{dv}{dt} \qquad (10.7)$$

This can be written as. If we call the left side of this equation position acceleration, then position acceleration and time acceleration are equal. This can also be said to be natural that the accelerations obtained are equal because the same phenomenon was observed using two different methods.

While the form of this equation is correct in itself, a mathematician might argue that this form is incorrect. They would say,

$$\frac{\partial(\frac{1}{2}v^2)}{\partial s} = \frac{\partial v}{\partial t}$$

It should be written as above, or so one might say. We will discuss the meaning of this strange symbol ∂ later, and for now, let's proceed using the form of equation (10.7).

Multiplying both sides of equation (10.7) by the mass m,

$$m\frac{d(\frac{1}{2}v^2)}{ds} = m\frac{dv}{dt} \qquad (10.8)$$

It can be written as. Newton's second law of motion is $F = m\frac{dv}{dt}$. Therefore, equation (10.8) is,

$$F = m\frac{d(\frac{1}{2}v^2)}{ds} \qquad (10.9)$$

This can be written as. This equation is an equivalent expression of force to Newton's second law of motion. The advantage of this expression is that acceleration can be defined as a function of position in space, which is a feature not found in time acceleration. In other words, it allows us to discuss the acceleration acting on a stationary object.

The well-known quantity of gravitational acceleration g is a function that depends on spatial position, and therefore can be considered a type of positional acceleration. Equation (10.8) answers the question of why we can set mg = ma.

10.4 Algebraic Derivation

Equation (10.7) can also be derived algebraically. The following equation seems completely meaningless.

$$\int \mathrm{vdv} = \int \mathrm{vdv}$$

This can be derived by transforming the equation. Since we can set v=$\frac{ds}{dt}$,

$$\int \mathrm{vdv} = \int \frac{\mathrm{ds}}{\mathrm{dt}}\mathrm{dv}$$

If it transforms,

$$\frac{\int vdv}{ds} = \frac{dv}{dt}$$

$$\int \mathrm{vdv} = \frac{1}{2}\mathrm{v}^2$$

$$\frac{d(\frac{1}{2}v^2)}{ds} = \frac{dv}{dt}$$

Thus, we obtained equation (10.7). Mathematics is a convenient thing. What I had struggled to figure out before, I was able to find the answer in less than a minute. However, considering its geometric meaning is not meaningless, so drawing many graphs was probably not meaningless either.

10.5 Velocity as a function of position

Rewriting equation (10.6),

$$a = \frac{du}{ds} \qquad (10.10)$$

However, $u = \frac{1}{2}v^2$

This can be written as. This u is a kinetic quantity as a function of position, and we will call it position velocity. From equation (10.10), immediately,

$$u = \int ads \qquad (10.11)$$

It can be written as. Since the spatial derivative of position velocity is position acceleration, it is natural that the spatial integral of position acceleration is position velocity.

10.6 Representation in Three Dimensions

Up until now, we have considered motion in one dimension, but this concept can, of course, also be expressed in three-dimensional space. All that is needed is to write three equations. The positional acceleration observed from each direction of motion in three-dimensional space is:

$$a_x = \frac{du_x}{dx} \tag{10.12}$$

$$a_y = \frac{du_y}{dy}$$

$$a_z = \frac{du_z}{dz}$$

$$\text{which } u = f(u_x, u_y, u_z)$$

Therefore, the three-dimensional positional acceleration $\mathbf{a}$ is,

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{10.13}$$

However, $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the x, y, and z directions, respectively.

Therefore, substituting equation (10.12),

$$\mathbf{a} = \frac{du_x}{dx}\boldsymbol{i} + \frac{du_y}{dy}\boldsymbol{j} + \frac{du_z}{dz}\boldsymbol{k} \tag{10.14}$$

Therefore, mathematicians perform calculations like this.

$$\mathbf{a} = \frac{du_x}{dx}\boldsymbol{i} + \frac{du_y}{dy}\boldsymbol{j} + \frac{du_z}{dz}\boldsymbol{k} = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k} \tag{10.15}$$

This was defined as. In mathematics, this type of calculation is called partial differentiation. Equation (10.15) is,

$$\mathbf{a} = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k} \quad = \mathrm{grad\,u} \qquad (10.16)$$

It is sometimes written as such.

If we extend equation (10.7) to three dimensions using this notation,

$$\text{grad}\mathrm{u} = \frac{d\boldsymbol{v}}{dt}$$

It can be written as, and equation (10.8) is,

$$\mathrm{m}\,\text{grad}\mathrm{u} = \mathrm{m}\frac{d\boldsymbol{v}}{dt}$$

It can be written as. If written in a way that appeals to mathematicians,

$$\text{grad}\ \mathrm{u} = \frac{\partial \mathbf{v}}{\partial \mathrm{t}} \qquad (10.19)$$

$$\mathrm{m}\ \text{grad}\ \mathrm{u} = \mathrm{m}\frac{\partial \mathbf{v}}{\partial \mathrm{t}} \qquad (10.20)$$

10.7 Difference between Ordinary and Partial Differentials

In mathematics, a differential equation involving one variable is called an ordinary differential equation. For example,

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{a}$$

It is written as. Differential equations with two or more variables are called partial differential equations, for example,

$$\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k}=\boldsymbol{a} \qquad (10.22)$$

It is written like this. The differential symbol is d for one variable and ∂ for two or more variables. Why is this distinction necessary?

One possible reason is that by using ∂ instead of d, it can be immediately determined that the differential equation is a differential equation of two or more variables, but equation (10.22)

$$\frac{du}{dx}\mathbf{i}+\frac{du}{dy}\mathbf{j}+\frac{du}{dz}\mathbf{k}=\mathbf{u}$$

If you write it out like this, you'll see dx, dy, and dz, and you can see that it's a differential equation with three variables without even needing to use ∂.

Furthermore, one could argue that this difference in notation is necessary because the derivative of one variable and the derivative of two or more variables are fundamentally different, and it is necessary to demonstrate this difference. Indeed, the derivative of one variable and the derivative of two variables are different, but the derivative of two variables and the derivative of three variables are also different. If it is necessary to demonstrate the difference between the derivative of one variable and the derivative of two variables, then it is also necessary to demonstrate the difference between the derivative of two variables and the derivative of three variables. For example, the symbol for the derivative of two variables should be ∂, and the symbol for the derivative of three variables should be δ. Since this has not been done, it is likely that there is basically no difference between the derivative of two variables and the derivative of three variables, which is why the same symbol ∂ can be used. On the other hand, how should we interpret the fact that the symbols used for the derivative of one variable and the derivative of two or more variables are different? The derivative of one variable is a special case of the derivative of two or more variables, and using a set expression, the derivative of one variable is included in the derivative of two or more variables. In fact, if the motion

represented by equation (10.22) is linear, then if it moves diagonally with respect to each of the arbitrarily chosen x, y, and z coordinate axes, then $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial z}$ are not zero, If we set up the coordinate system so that the motion is parallel to the z-axis, then $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ becomes zero, and equation (10.22) becomes:

$$0 + 0 + \frac{\partial u}{\partial z}\boldsymbol{k} = \frac{\partial u}{\partial z}\boldsymbol{k} = \boldsymbol{a}$$

This is the result. While writing it this way is mathematically correct, according to the conventions of mathematicians, since this equation is a differential equation of one variable, it is incorrect to use the partial derivative notation.

$$\frac{du}{dz}\boldsymbol{k} = \boldsymbol{a}$$

This is how it must be written. Such meaningless complexity only makes the interpretation of mathematical formulas unnecessarily difficult and should be eliminated from the world of mathematics. The differential symbol should be either d or ∂. Basically, the differential symbol is d, so this symbol should be used. It seems that in the past, this symbol was standardized, but I don't know who first suggested it, but it seems that this person's shallow idea and the enjoyment of mathematicians at the time who took pleasure in complicating things led to the introduction of the strange symbol ∂ into the world of mathematics. Mathematicians sometimes introduce such strange conventions.

For example, this also applies to the conventions for using fractions ($13\frac{2}{3} \neq 13 \times \frac{2}{3}$, $13\frac{2}{3} = 13 + \frac{2}{3}$) and parentheses (〝（〞, 〝｛〞, 〝［〞).

In modern times, the differential symbol d is used for single-variable differential equations, and ∂ is used for differential equations with two or more variables. We are not obligated to follow this foolish convention, but considering its affinity with the knowledge of physics that adheres to this convention, we will reluctantly follow it. In the mathematical transformations described

below, please do not be surprised if the differential symbol changes from d to ∂ or from ∂ to d. For example, equation (10.7)

$$\frac{d(\frac{1}{2}v^2)}{ds} = \frac{dv}{dt}$$

Even if we write it like this, we don't want this notation to be judged as incorrect, and after writing such an expression, that is,

$$\frac{\partial(\frac{1}{2}v^2)}{\partial s} = \frac{\partial v}{\partial t}$$

Even if we write things like "that is," we don't want people to think of us as wizards.

The decision of whether to use 'd' or '∂' as the differential symbol is simply based on whether the equation is a single-variable differential equation or a two-variable differential equation. This is merely a convention and has no mathematical significance whatsoever. It is perfectly acceptable to use '∂' for a single-variable differential equation, and it is not incorrect to use 'd' for a two-variable differential equation.

10.8 Positional Acceleration and Hamilton's Principle

Those with a somewhat advanced level of knowledge, such as Hamilton's principle or the principle of least action, might argue that Hamilton was the first to obtain a result like equation (10.10). While I don't intend to discuss Hamilton's principle here, his ideas require concepts like kinetic energy and positional energy, employing a much more complex framework than our concept of positional acceleration. Our concept of positional acceleration, derived purely from considering the concept of acceleration, is fundamentally unrelated to kinetic energy or potential energy, and can be considered a more basic concept than Hamilton's.