Chapter 7: Motion

~Basic Concepts of Differential and Integral Calculus~

For those who have an allergic reaction to the mere mention of differential and integral calculus, this chapter will explain the concept using the concrete example of motion. While I have tried to alleviate the allergy to differential and integral calculus as much as possible, it is quantitatively impossible to provide a systematic and sufficient explanation of this major field of mathematics in this chapter. Therefore, it may have worsened your allergy. In that case, I hope you will supplement your knowledge of differential and integral calculus with a suitable mathematics textbook.

This chapter will be fundamentally mathematical, and we will not consider the subtleties of physical time and space, assuming that the flow of time and the scale of space remain constant and unchanging.

7.1 Description of the phenomenon

Generally, the state of an object changes in various ways over time. The fundamental task of physics is to discover the laws that govern these changes in objects, but to do so, there must be a way to describe and record these changes. First, a method is needed to quantitatively describe the changes in objects. In general, it may be possible to quantitatively describe all phenomena, but it is not possible to quantitatively measure all phenomena. For example, it is almost impossible with modern technology to quantitatively measure changes in a person's mind, and it is also difficult to measure the changes in objects related to collisions that occur in extremely short periods of time. In other words, in order to quantitatively describe the changes in objects, some kind of measurement must be performed, and if the accuracy of this measurement is not sufficient, it is not possible to quantitatively and accurately describe the changes in objects. If the changes in objects cannot be quantitatively described, it is not possible to discover the laws that govern them, so in physics, it seems that only those changes that can be quantitatively described at that point in time are dealt with.

Furthermore, there are many cases where the amount of change in an object of interest cannot be measured directly, but indirect quantities can be measured. For example, to measure the strength of a gravitational field, one can place an object into the gravitational field and measure the acceleration acting on it. However, what we really want to know is what is accelerating the object. Yet, no one has directly measured this quantity. Currently, it is not possible to directly measure changes in a person's mind, but by measuring changes in the amount of electricity flowing through electrodes attached to the body, as in a lie detector, we can infer that some kind of change has occurred in that person's mind. Such methods are also attempts to measure indirectly.

Generally, measurement involves measuring several physical quantities that correspond to each other. Mathematically, the relationships between these physical quantities can be described as functional relationships.

There are three important ways to write functions:

(1) Table representation

This is a function representation obtained by recording the measured data in a tabular format. This method is often used during the measurement phase.

(2) Graph representation

Representing measured data in graph format is an effective way to understand its geometric characteristics. Furthermore, seismometers, for example, directly acquire measured data in graph format.

(3) The performance of the analysis

When a functional relationship is expressed by an equation, it is said to be expressed analytically. Since there is a one-to-one

correspondence between the equation of a function and its graph, analytical expressions can often be obtained directly from the graphical representation of a function. This method requires knowledge of the shape of the function's graph. When a function obtained through experimental measurement is expressed analytically, it constitutes the discovery of a law. Laws obtained in this way are called experimental laws. There are many such laws, and this can be considered a fundamental method of law discovery.

Even in cases where a function cannot be expressed analytically as a formal representation,

$$y = f(x) \quad \text{or} \quad F(x,y) = 0$$

Functions are sometimes expressed in this way.

7.2 Basic Concepts of Differentiation

How can we describe the motion of an object? To describe the position of an object, we need three points because our space is three-dimensional. For example, the position in Descartes' coordinate system is, if you give three points such as (x1, y1, z1), It is possible to describe the relative position from the origin of the reference coordinate system.

To describe the time corresponding to the position of an object, one can measure the elapsed time from an arbitrary reference time.

To describe a position in three-dimensional space, three points are necessary, but since three-dimensional space cannot be drawn on two-dimensional paper, it is impossible to geometrically represent the relationship between space and time. Pictures that appear three-dimensional are illusions and not reality. Geometric representation is a useful method because it allows us to understand phenomena visually. If the motion is one-dimensional in space, then together with time it is two-dimensional, and can be drawn on paper. To make the phenomenon easier to understand, let's consider one-dimensional motion, such as motion along a straight line, for a while. This one-dimensional motion is not useless when considering three-

dimensional motion, because three-dimensional motion can be divided and represented as three one-dimensional motions.

Now, let's assume there is a car moving along a straight line, and we will simultaneously record the distance s and time t traveled by this car. In this case, there is a functional relationship between s and t.

$$s = f(t)$$

It can be written formally as above.

The graph of the special case of this function, where s and t are proportional, is the graph of a so-called linear function, and is the graph of uniform linear motion.

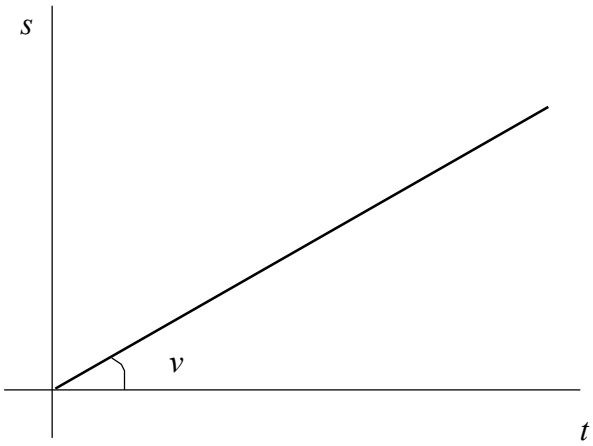


Figure 7.1 Proportional relationship between distance and time

If we denote the slope of this line as v, then the analytical representation of the graph with initial values of s and t set to zero is:

$$s = vt \quad \text{or} \quad v = \frac{s}{t} \tag{7.1}$$

This can be written as. This v is the ratio of the change in distance s to time t. A larger v indicates a larger change in s over time, and if v is zero, the change in s with respect to t is also zero. This ratio v represents the change in distance with respect to time t, and such a quantity is called velocity. Note that the variables distance and time here are not place and time. Later we will consider infinitesimal distances and infinitesimal times, but the concepts of infinitesimal place and infinitesimal time are completely meaningless for point places and time. This relationship in equation (7.1) cannot be applied to the general case where time and distance are not proportional, that is, when v is not constant. This is because the change in distance is not constant with respect to the change in time. How can we apply this to the general case? A typical graph of distance s and time t can be represented, for example, as shown in Figure 7.2.

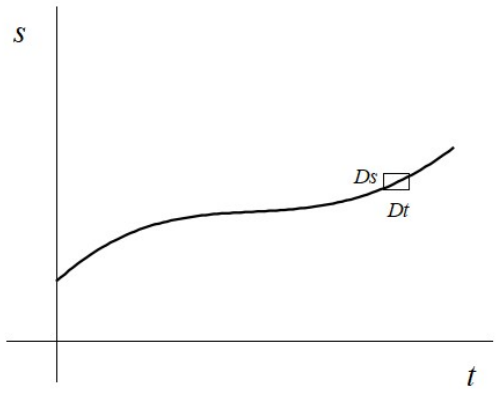


Figure 7.2 Relationship between distance and time

If we magnify a point on this graph as if viewing it under a microscope, and increase the magnification, we can consider this graph to be partially a straight line (Figure 7.3).

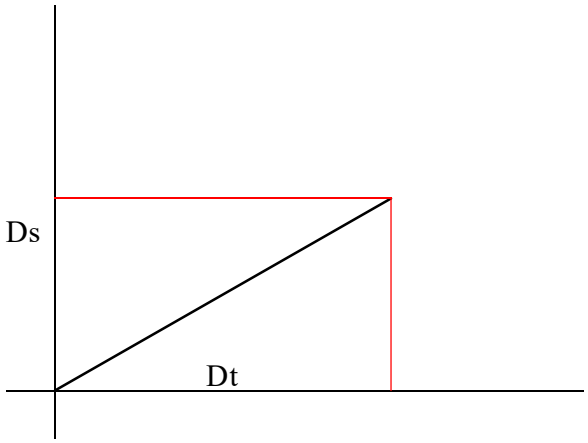


Figure 7.3 Proportional relationship between minute distances and minute time intervals

This is similar to saying that although the Earth's surface is spherical, to us, who are sufficiently small relative to the Earth, it can be considered a plane. Since such a local graph can be considered to be in the same form as a graph like the one in Figure 7.1, the velocity at that point can be given by, if $\mathrm{D}s$ is the infinitesimal distance and $\mathrm{D}t$ is the infinitesimal time,

$$\mathrm{D}s = v\mathrm{D}t \tag{7.2}$$

It can be written

By performing this operation at each point on the graph, we can determine the velocity, which is the slope at each point.

If Dt approaches zero indefinitely, then equation (7.2) holds strictly. In mathematical terms,

$$v = \lim_{Dt \to 0} \frac{Ds}{Dt} = \frac{ds}{dt} \tag{7.3}$$

This quantity $\frac{ds}{dt}$ is called the derivative of s with respect to t, and the calculation of this is called differentiation. Sometimes $\frac{ds}{dt}$ is written as s' , $f'(t)$, as $\frac{d}{dt}f(t)$, etc. dt is a quantity that approaches zero indefinitely, and if we call this the infinitesimal time, then the relationship between this infinitesimal time dt and the infinitesimal distance ds that corresponds to it is,

$$ds = vdt$$

This can be written. This equation holds strictly.

7.3 Function and Increment

A function consisting of one variable is,

$$s = f(t) \qquad (7.5)$$

I was able to write about it abstractly.

If we define a quantity called the increment of a variable,

$$\Delta t = t_2 - t_1 \qquad (7.6)$$

t_2, t_1 is one value of t

We can write it, and since the quantity of the increment of this variable also satisfies equation (7.5), if we write equation (7.5) using this quantity,

$$\Delta s = f(\Delta t)$$

It can be written. If we add this equation and equation (7.5) together piece by piece,

$$s + \Delta s = f(t + \Delta t) \qquad (7.7)$$

This formal functional relationship can be obtained.

7.4 Basic Techniques for Differential Calculus

Let's use the relationship in equation (7.7) to perform a differential calculation using a function as an example. Suppose that s and t are related by the following functional relationship.

$$s = f(t) = at^2 + bt + c \qquad (7.8)$$

Let's find $\frac{ds}{st}$ at this time.

Equation (7.8) can be rewritten from equation (7.7) as follows:

$$\begin{aligned} s + \Delta s &= a(t + \Delta t)^2 + b(t + \Delta t) + c \\ &= at^2 + bt + c + b\Delta t + 2at\Delta t + a(\Delta t)^2 \end{aligned}$$

Here, since $s = at^2 + bt + c$

$$\Delta s = b\Delta t + 2at\Delta t + a(\Delta t)^2$$

$$\frac{\Delta s}{\Delta t} = b + 2at + a\Delta t$$

This can be written. Since the limit of this Δt as it approaches 0 is the derivative,

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = b + 2at$$

This is how the derivative can be calculated.

7.5 Definition of Differential

Let's derive the well-known definition of differentiation from equation (7.7). From equation (7.7),

$$\Delta s = f(t + \Delta t) - s$$

However,since s=f(t),

$$\Delta s = f(t + \Delta t) - f(t)$$

Divide both sides by Δt,

$$\frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

From the definition of equation (7.3),

$$\frac{d\,s}{d\,t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} \qquad (7.9)$$

Thus, the definition of differentiation was derived.

7.6 Differential Formulas

Here, we will list the main formulas, such as the derivative calculation results obtained using the method in 7.4.

Functions	derivatives		
$y = kf(x)$	$y' = kf'(x)$	where k is constant	(7. 10)
$y = f(x) + g(x)$	$y' = f'(x) + g'(x)$		
$y = f(x) - g(x)$	$y' = f'(x) - g'(x)$		
$y = f(x)g(x)$	$y' = f'(x)g(x) + f(x)g'(x)$		
$y = \frac{f(x)}{g(x)}$	$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$		
$y = k$	$y' = 0$	where k is constant	
$y = x^a$	$y' = ax^{a-1}$	where k is constant	
$y = e^x$	$y' = e^x$	where $e = \lim_{t\to\infty}(1 + \frac{1}{t})^t$	
$y = \log\|x\|$	$y' = \frac{1}{x}$		
$y = \sin x$	$y' = \cos x$		
$y = \cos x$	$y' = -\sin x$		
$y = \tan x$	$y' = \frac{1}{(\cos x)^2}$		

7.7 Differentiation in Three Dimensions

The one-dimensional velocity could be expressed as shown in equation (7.3). The velocity of an object moving in three-dimensional space can be expressed by simply writing three of these one-dimensional velocities. The velocity v_x obtained by measuring the change in the x-coordinate of the object with respect to time t is:

$$v_x = \frac{dx}{dt}$$

Similarly, in y, z coordinates,

$$v_y = \frac{dy}{dt}$$

$$v_z = \frac{dz}{dt}$$

It can be written, the velocity v is

$$v = f(v_x, v_y, v_z)$$

Therefore, if we find v_x, v_y, and v_z individually, v can be determined. If we write this velocity v in vector form,

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

And the |v| magnitude of v is,

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

That is the case.

7.8 Differentiation of a Vector

The position vector is expressed as $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$, where rx is the value of the x-coordinate.Since it represented that, if we write r_x in terms of x, the position vector is,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

This can be written. If we assume that vector $\mathbf{r}$ is a function of time t and is differentiable with respect to t, then $\frac{dx}{dt}$ is the time derivative of the x component of vector $\mathbf{r}$. Multiplying this component by the unit vector i, the quantity $\frac{dx}{dt}\boldsymbol{i}$ is the time derivative vector of the x component of vector r. Similarly, the y and z components are $\frac{dy}{dt}\boldsymbol{j}$ and $\frac{dz}{dt}\boldsymbol{k}$, and the sum of these three vectors can be considered as the time derivative vector of vector $\mathbf{r}$.

If we write this vector as r',

$$\mathbf{r'} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

Therefore, if we write this $\mathbf{r'}$ as $\frac{dr}{dt}$,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \mathbf{v} \qquad (7.11)$$

It can be written as. If we separate a vector into x, y, and z components, they become scalars, and if we multiply each scalar by the unit vector of the corresponding component, we get a vector in that direction. Differentiation of a vector can be reduced to differentiation of a scalar, and the result of that calculation can be converted back to a vector, so all differentiation formulas for scalars can also be applied to vectors.

7.9 Basic Concepts of Definite Integrals

The velocity is given by equation (7.3),

$$v = \frac{ds}{dt} \qquad \text{again}(7.3)$$

It can be written, but in differentiation we found v from s and t. Conversely, how can we find s when v is known? According to the approximate expression of equation (7.2),

$$Ds = vDt \qquad \text{again}(7.2)$$

We were able to write this. From this, it is immediately clear that the infinitesimal distance Ds is the product of the infinitesimal time Dt and the corresponding time v, so by adding this over the interval in which we want to find the infinitesimal distance, we can find the distance s. The interval we want to find is, for example, a certain time interval Δt, and we will find the distance Δs that moved during this time (Figure 7.4).

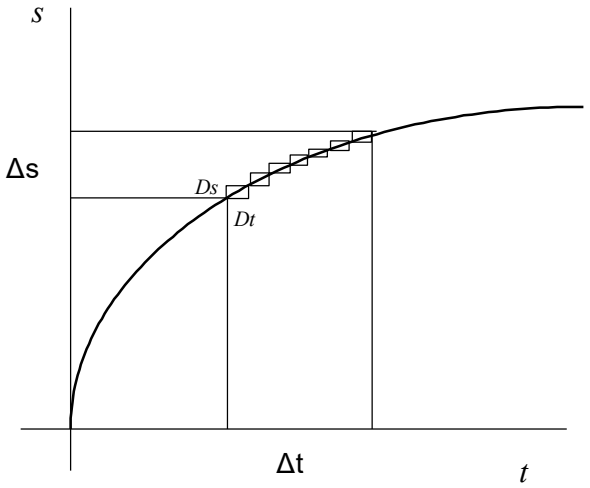


Figure 7.4 Sum of infinitesimal time and sum of infinitesimal distance

Since Δt can be divided into several infinitesimal time intervals, if we divide it into n infinitesimal time intervals, then Δt becomes

$$\Delta t = Dt_1 + Dt_2 + ... + Dt_n \qquad (7.12)$$

It can be written as. The velocity v, which is the slope corresponding to each infinitesimal time, is given in this case, and since v can be considered a function of t, if we write it as v(t), then the product of this v(t) with the corresponding infinitesimal time corresponds to each infinitesimal distance. Therefore, the distance Δs is the sum of these,

$$\Delta s = v(t_1)Dt_1 + v(t_2)Dt_2 + ... + v(t_n)Dt_n \qquad (7.13)$$

This allows us to find it. If we assume that the intervals between each infinitesimal time are equal, the larger the number of divisions n, the more accurately we can find Δs. This is because, as with differentiation, increasing the number of divisions improves the accuracy of the approximation in equation (7.2). In other words, to find Δs accurately, we can use $n \to \infty$,

$$\Delta s = \lim_{n\to\infty} v(t_1)Dt_1 + v(t_2)Dt_2 + ... + v(t_n)Dt_n \qquad (7.14)$$

This is how it works. Using this formula, even a complex function v(t) that cannot be expressed analytically (for example, data obtained from a graph) can be obtained by giving t v(t), and by giving appropriate conditions such as $n \to 100$ or $n \to 1000$ and performing tedious calculations, Δs can be approximately found. In mathematics, this type of calculation is called integration, and if we write the time interval Δt as $\Delta t = t_2 - t_1$, and the distance corresponding to that time as $\Delta s = s$, then calculations like those in equation (7.14) are performed in mathematics,

$$s = \int_{t_1}^{t_2} v(t)dt \qquad (7.15)$$

This is written as such, and this type of calculation is called a definite integral.

7.10 Relationship between Antiderivatives and Derivatives

The function before its derivative is derived is called the antiderivative. Let's consider the relationship between this antiderivative and the derivative. For example, distance is a function of time.

$$s = f(t)$$

We can consider this to be the case, but the derivative obtained by differentiating s with respect to t is the velocity,

$$v(t) = \frac{ds}{dt} = \frac{df(t)}{dt} = f'(t)$$

This can be written as. In this case, f(t) is the antiderivative of v(t). The v(t) obtained in this way represents the slope of the graph of s = f(t) at each time point. If we plot a graph with t on the horizontal axis and v(t) on the vertical axis, this graph will be a graph of the slope corresponding to each t. As shown in equation (7.2), the product of this slope v(t) and the infinitesimal time Dt is an finitesimal distance Ds, and the product of v(t) and Dt is the area of the rectangle with those sides, so this infinitesimal distance Ds is the area of the rectangle formed by v(t) and Dt. Just as in 7.9, Δt can be expressed by equation (7.12) and Δs can be written by equation (7.13), so we can obtain equation (7.14) in exactly the same way. Therefore, the geometric meaning of integrating the function v(t) is to find the area of the region enclosed by the function v(t) and the t-axis, and that area represents the distance s (Figure 7.5).

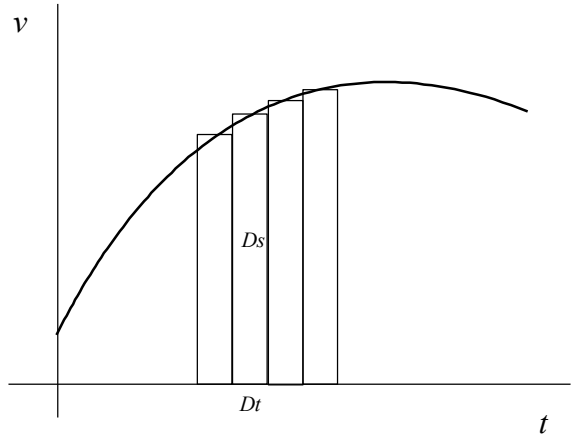


Figure 7.5 Geometric meaning of integrals

7.11 Calculation of Indefinite Integrals

The antiderivative is

$$s = f(t)$$

The derivative v(t) obtained by differentiating with respect to t in the case given by is:

$$v(t) = \frac{ds}{dt} = \frac{df(t)}{dt} = f'(t)$$

However, to perform an indefinite integral of the derivative v(t) is to perform the reverse calculation of differentiation, which means to find the antiderivative. Indefinite integral is,

$$\int f'(t)dt = \int \frac{ds}{dt}dt = \int \frac{df(t)}{dt}dt = f(t) \qquad (7.16)$$

It is defined as performing the calculation of . For example, if the antiderivative is,

$$s = f(t) = t^2$$

If so, the derivative v(t) obtained by

$$v(t) = \frac{ds}{dt} = \frac{d(t^2)}{dt} = 2t$$

Therefore, integrating this v(t) yields the antiderivative,

$$\int v(t)dt = s = f(t) = t^2$$

It can be calculated. In this case, v(t) = 2t,Therefore, integrating this v(t) yields the antiderivative,

$$\int 2t dt = t^2$$

And, 2t that integrating with respect to t gives t². Basic integral calculations are derived using this kind of inverse method. However, since differentiating a constant results in 0, the calculation of the integral is not uniquely determined in general. For example, if the antiderivative is,

$$s = t^2 + c \qquad \text{where c is an arbitrary constant.}$$

If so, then differentiating this function gives,

$$\frac{ds}{dt} = 2t$$

Therefore, when we integrate this function,

$$\int 2t dt = t^2 + c$$

This must be the case. This c is called the integration constant. Physically, the integration constant is determined by an arbitrary condition called the setting of the initial value. For example, in experiments, it is up to the observer to decide whether to set the initial value of the time measured to the current time and always record it at the time being measured, or to use a stopwatch and set the initial value to zero before recording. Since the result will naturally change depending on how this initial value is set, it is natural that such an arbitrary value will appear mathematically as well.

7.12 Relationship between Definite and Indefinite Integrals

As a special form of definite integral, the definite integral from t = c (where c is an arbitrary constant) to the variable t is:

$$s = \int_c^t v(t)dt \qquad (7.17)$$

This can be written as, but if the interval$[c, t]$ is continuous, then this definite integral can be considered an indefinite integral. That is,

$$s = \int_c^t v(t)dt = \int v(t)dt \qquad (7.18)$$

It can be written. From the definition of the definite integral,

$$\int_c^t v(t)dt = \int_c^a v(t)dt + \int_a^t v(t)dt \qquad \text{a is any constant}$$

It is immediately clear that the formula holds. This is because we have simply divided the interval of the definite integral. Since a is an arbitrary constant, if we set it to 0,

$$\int_c^t v(t)dt = \int_c^0 v(t)dt + \int_0^t v(t)dt \qquad (7.19)$$

This can be written as. The first term on the right-hand side of this equation is a definite integral with respect to an arbitrary constant, and since the interval$[c, 0]$ is continuous, it can be calculated to be an arbitrary constant. If we set this constant to C,

$$\int_c^t v(t)dt = C + \int_0^t v(t)dt \qquad (7.20)$$

This can be written. In other words, the indefinite integral can be expressed as the sum of an arbitrary constant C and a more restrictive second term on the right-hand side.

The integral constant does not appear in the calculation result of the second term on the right-hand side. For example, if we calculate using v(t) = 2t,

$$\int_0^t 2t\,dt = [t^2]_0^t = t^2$$

This is the result. Since this result is the same as the indefinite integral with the integration constant omitted, the mathematical meaning of this form is a definite integral from 0 to the variable t. Such an integral is an intermediate between definite and indefinite integrals, and we will call it a limited integral. Since the notation as in equation (7.20) is somewhat cumbersome, we will use the following notation:

$$\int_0^t v(t)\,dt \equiv \int_0 v(t)\,dt \qquad (7.21)$$

Let's write it like this. In this book, further,

$$\int_0 v(t)\,dt \equiv \int v(t)\,dt \qquad (7.22)$$

It is sometimes written as such.

Because an integration constant is involved in indefinite integrals, it cannot be strictly said that they are the inverse of differentiation. For example, differentiating t^2 always gives 2t, but the indefinite integral of 2t is t^2 + c, which is not equal to t^2. On the other hand, if we perform a limited integral of 2t, we always get t^2, so we can strictly say that limited integrals are the inverse of differentiation. This lack of arbitrariness is convenient in that it allows us to express things clearly. However, there is a constraint that the interval$[0, t]$ must be continuous.

7.13 Integral of Zero

Integration is the inverse of differentiation, but there are exceptions to this concept. Differentiating any constant yields 0, but what happens when you integrate 0? If we consider integration to be the inverse of differentiation, then integrating 0 should yield any constant, but strangely, integrating 0 does not yield any constant. The fact that the derivative is 0 means, for example, that a function of velocity can be expressed as v(t) = 0. If we plot time t on the horizontal axis and v(t) on the vertical axis, the graph of v(t) = 0 is a straight line that coincides with the t-axis.

Therefore, since v(t) is zero everywhere in t, when we integrate this v(t) = 0 over any interval in t, the area of the region enclosed by v(t) and the t-axis is 0. Thus,

$$\int 0\mathrm{dt} = 0 \qquad (7.23)$$

Therefore, the integral of 0 is 0.

7.14 Integral Formulas

Let's summarize the integral formulas obtained using methods such as those described in section 7.11.

An indefinite integral can be expressed as the sum of a limiting integral and an arbitrary constant. If the interval$[0, \mathrm{x}]$ are continuous,

$$\int f(x)dx = \int_0 f(x)dx + c \quad \text{(where c is an arbitrary constant)} \qquad (7.24)$$

It can be written. Adding c to the result of the limited integral gives the indefinite integral.

The main formulas for calculating indefinite integrals are listed below. The integration constant is omitted.

Functions	derivatives	
$y = 0$	$\int 0dx = 0$	(7.25)
$y = k$	$\int kdx = kx$	where k is constant
$y = x^a$	$\int x^a dx = \frac{x^{a+1}}{a+1}$	a is constant of a $\neq$-1
$y = x^{-1} = \frac{1}{x}$	$\int x^{-1}dx = \log \lvert x \rvert$	
$y = e^{ax}$	$\int e^{ax}dx = \frac{1}{a}e^{ax}$	a is constant of a $\neq 0$
$y = \log \lvert x \rvert$	$\int \log \lvert x \rvert dx = x \log \lvert x \rvert - x$	
$y = \sin ax$	$\int \sin axdx = -\frac{1}{a}\cos ax$	
$y = \cos ax$	$\int \cos axdx = \frac{1}{a}\sin ax$	

$$\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$$

$$\int af(x)dx = a\int f(x)dx \quad \text{where a is constant}$$

The following formula can be immediately obtained from the definition of the definite integral, where the intervals $[a, b]$ are assumed to be continuous.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_b^a \{f(x) \pm g(x)\}dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$$

$$\int_b^a kf(x)dx = k\int_b^a f(x)dx \quad \text{where k is a constant}$$

$$\int_b^a f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{where c is an arbitrary constant}$$

7.15 Improper Integrals

The definite integrals discussed so far can only be calculated if the integration interval is continuous. Here, we extend the concept of definite integrals to other cases. A function is continuous on the integration interval if $\mathrm{v}(\mathrm{t})$can be represented by a continuous curve. For example, in the case of $\mathrm{v}(\mathrm{t})$=1/t, it is not continuous on the interval$[-1,\ 1]$. This is because $\mathrm{v}(\mathrm{t})$ becomes indeterminate at t=0.

Such integrals are calculated by dividing the interval and summing the definite integrals over continuous intervals. Integrals calculated in this way are called improper integrals. There are four types of integrals that can be divided:

Cases where v(t) is continuous in (a, b]

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{v}(\mathrm{t})\mathrm{dt} = \lim_{\varepsilon \to +0} \int_{a+\varepsilon}^{b} v(t)dt \quad (7.26)$$

Cases where v(t) is continuous in [a, b)

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{v}(\mathrm{t})\mathrm{dt} = \lim_{\varepsilon \to +0} \int_{a}^{b-\varepsilon} v(t)dt$$

Cases where v(t) is continuous in [a, +∞)

$$\int_{a}^{+\infty} v(t)dt = \lim_{b \to +\infty} \int_{a}^{b} v(t)dt$$

Cases where v(t) is continuous in (-∞, b]

$$\int_{-\infty}^{b} v(t)dt = \lim_{a \to -\infty} \int_{a}^{b} v(t)dt$$

However, expressions like $(a, b]$ indicate an interval that does not include a.

In these integrals, if a limit exists, the integral is said to converge; if a limit does not exist, it is said to diverge.

The integration interval is divided into, for example, interval$[-1, 1]$, which does not contain zero, and then calculated by dividing it into intervals $[-1, 0)$ and $(0, 1]$.

$$\int_{-1}^{1} v(t)dt = \lim_{\varepsilon \to +0} \int_{-1}^{-\varepsilon} v(t)dt + \lim_{\varepsilon \to +0} \int_{\varepsilon}^{1} v(t)dt$$

The calculation is performed in this manner. For example, the improper integral of section $[-1, 1]$ in the case of $v(t) = 1/t$ is:

$$\int_{-1}^{1} \frac{1}{t}dt = \lim_{\varepsilon \to +0} \int_{-1}^{-\varepsilon} \frac{1}{t}dt + \lim_{\varepsilon \to +0} \int_{\varepsilon}^{1} \frac{1}{t}dt$$

It is said that... Regarding such integrals, mathematicians (though perhaps not all) think as follows: Calculating the right-hand side of this equation,

$$\lim_{\varepsilon \to +0} \int_{-1}^{-\varepsilon} \frac{1}{t}dt = \lim_{\varepsilon \to +0} [\log|t|]_{-1}^{-\varepsilon} = \lim_{\varepsilon \to +0} \log\varepsilon = -\infty$$

$$\lim_{\varepsilon \to +0} \int_{\varepsilon}^{1} \frac{1}{t}dt = \lim_{\varepsilon \to +0} [\log|t|]_{\varepsilon}^{1} = \lim_{\varepsilon \to +0} (-\log\varepsilon) = +\infty$$

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And adding these two together gives the integral over the entire interval,

$-\infty + \infty$ =undefined

Therefore, it is concluded that this integral cannot be a constant value. We are not very concerned with such conclusions. This integral should be zero. Let's prove this.

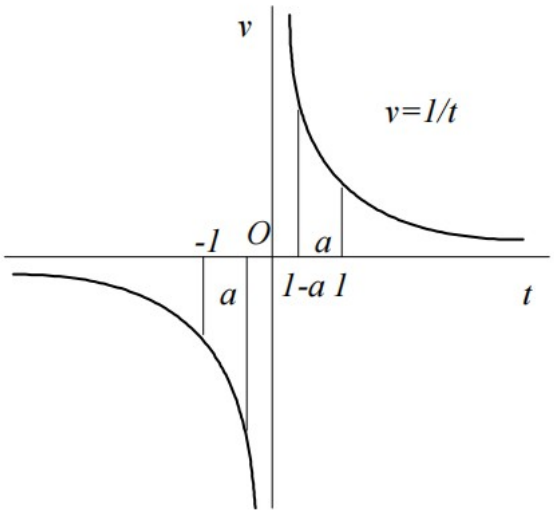


Figure 7.6 Improper integrals

If a has a value of $0 < a < 1$, then the definite integrals over the intervals$[-1, 0)$ and $(0, 1]$ that do not contain 0 are, respectively,

$$\int_{-1}^{-1+a} \frac{1}{t} dt \qquad (7.27)$$

$$\int_{1-a}^{1} \frac{1}{t} dt = -\int_{1}^{1-a} \frac{1}{t} dt \qquad (7.28)$$

As is clear from the figure, the graph of 1/t is symmetrical with respect to the origin *O*, and the absolute value of the area given by equation (7.27) is equal to the absolute value of equation (7.28). Since equation (7.27) is negative, the sum of equation (7.27) and equation (7.28) is always 0. That is,

$$\int_{-1}^{-1+a} \frac{1}{\tau} dt - \int_{1}^{1-a} \frac{1}{\tau} dt = 0 \qquad (7.29)$$

Therefore, since a is arbitrary in the range 0 < a < 1, equation (7.29) also holds in the limit as a → 1. This limit becomes the integral over the interval [-1, 1],

$$\lim_{a \to 1} \left(\int_{-1}^{-1+a} \frac{1}{\tau} dt - \int_{1}^{1-a} \frac{1}{\tau} dt \right) = 0$$

Therefore, the integral over the interval[-1,1] is zero. Formulas related

$$\lim_{x \to a} \{f(x) \pm g(x)\} = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

This can be written if there are limits in $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$, and if it diverges,

$$\lim_{a \to 1} \left(\int_{-1}^{-1+a} \frac{1}{t} dt - \int_{1}^{1-a} \frac{1}{t} dt \right) = \lim_{a \to 1} \int_{-1}^{-1+a} \frac{1}{t} dt - \lim_{a \to 1} \int_{1}^{1-a} \frac{1}{t} dt \qquad \text{[incorrect]}$$

Writing it that way is incorrect. In general, if the intervals into which an improper integral is divided diverge, the sum of the divided intervals cannot be considered as the value of that improper integral.

7.16 Vector Integrals

Similar to the case of differentiation of a vector, a vector can be considered as a function of three scalars, and therefore the integral of a vector can be calculated in the same way as the integral of a scalar. For example, the integral of a vector $\mathbf{v}(t)$ with respect to t is:

$$\int \mathbf{v}(t)dt$$

It is written. If we set the vector $\mathbf{v}(t)$ as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$,

$$\begin{aligned}\int \mathbf{v}(t)dt &= \int(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k})dt \\ &= \int v_x\mathbf{i}dt + \int v_y\mathbf{j}dt + \int v_z\mathbf{k}dt\end{aligned}$$

i, **j**, and **k** can be considered constants,

$$= \mathbf{i}\int v_x dt + \mathbf{j}\int v_y dt + \mathbf{k}\int v_z dt$$

If we let $\mathbf{r} = \int \mathbf{v}(t)dt$, then $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$,

$$\begin{aligned}r_x &= \int v_x dt \\ r_y &= \int v_y dt \\ r_z &= \int v_z dt\end{aligned}$$

Thus, the integral of a vector can be reduced to a function of the integral of a scalar.