Chapter 6: Vectors

This chapter will discuss the basic definition of a vector. The content is mathematical and nothing new, so those who are already familiar with vectors do not need to read this chapter. However, the concept of vector quotients, which will be covered in this chapter, is often not found in general vector textbooks, so many people may not be familiar with this concept. Those people should read this chapter.

6.1 Scalars and Vectors

In physics, quantities can be broadly categorized into those that can only be expressed in terms of magnitude, such as length, time, and temperature, and those that possess both magnitude and direction, such as velocity and force. The former are called scalar quantities or simply scalars, while the latter are called vector quantities or vectors.

6.2 Representation of Vectors

A vector can be graphically represented by an arrow. The length of the arrow is the magnitude of the vector, and the direction of the arrow is the direction of the vector.

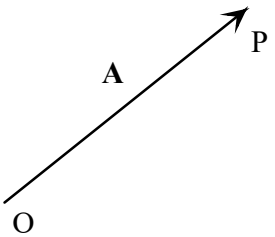


Figure 6.1 Graphical representation of a vector

As shown in Figure 6.1, when there is a vector from point O to point P, point O is called the starting point and point P is called the ending point, and algebraically it is written as $\overrightarrow{OP}$.

Alternatively, if we name this vector A, it can also be written as $\vec{A}$、$\bar{A}$、$\boldsymbol{A}$. In this book, we will write it in bold, like ***A.***

6.3 Composition of Vectors

As shown in Figure 6.2, when two vectors A and B act simultaneously on a single point O, and their combination is represented by vector C,

$$\mathbf{C = A + B} \qquad (6.1)$$

This is how it is written. This method of composition is called the parallelogram rule, or the triangle rule.

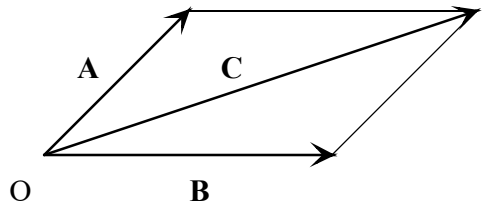


Figure 6.2 Synthesis of vectors

When applying this mathematics to physics, it is necessary to conduct experiments.If A and B are forces, then whether their resultant force is C can be determined by experimentation.It's something you can't know until you try it. Conversely, a vector is a quantity that has direction and whose composition follows the law of parallelograms. In physics, there is a principle called superposition, and quantities that can be combined in this way can be said to be superimposable quantities.

Vectors have magnitude and direction, but vectors with the same magnitude but opposite direction are represented by a negative sign. That is, a vector in the opposite direction to vector $\mathbf{A}$ is represented by $-\mathbf{A}$. This composition is:

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{A} - \mathbf{A} = \mathbf{O} \qquad (6.2)$$

This is the result. O is a formal vector with zero magnitude and is called the zero vector.

6.4 Product of Vectors and Scalars

If we combine vector $\mathbf{A}$ and vector $\mathbf{A}$ according to the parallelogram law, the resulting vector will be in the same direction as $\mathbf{A}$, but twice as large. Algebraically, this can be written as follows:

$$\mathbf{A} + \mathbf{A} = 2\mathbf{A} \qquad (6.3)$$

This is the result. In general, the product of a scalar n and a vector **A** is written as n**A**, where n**A** has the same direction as **A** and is n times the magnitude of **A**.

Since **A** vector can be considered as n times **s** with the same direction and unit length,

$$\mathbf{A} = n\mathbf{s} \qquad (6.4)$$

It can be written as above. **A** vector with a unit length is called a unit vector. In this case, n is equal to the magnitude of vector **A**, so if we write the magnitude of vector **A** as |A| or A, and rewrite it,

$$\mathbf{A} = |\mathbf{A}|\mathbf{s} = A\mathbf{s} \qquad (6.5)$$

It can be written as above.

6.5 Analytical Representation of Vectors

As shown in Figure 6.3, the vector **B** represented on a two-dimensional coordinate system is

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} \qquad (6.6)$$

$$B_x = B\cos(B,x)$$

$$B_y = B\cos(B,y)$$

i and **j** are vectors with unit lengths in the x and y directions, respectively.

It can be written as above. B_x is called the x-component of vector **B**. Vector **B** can be considered as a combination of $B_x\mathbf{i}$ and $B_y\mathbf{j}$ according to the parallelogram law. The magnitude of vector **B** is,

$$|\mathbf{B}| = B = \sqrt{B_x^2 + B_y^2} \qquad (6.7)$$

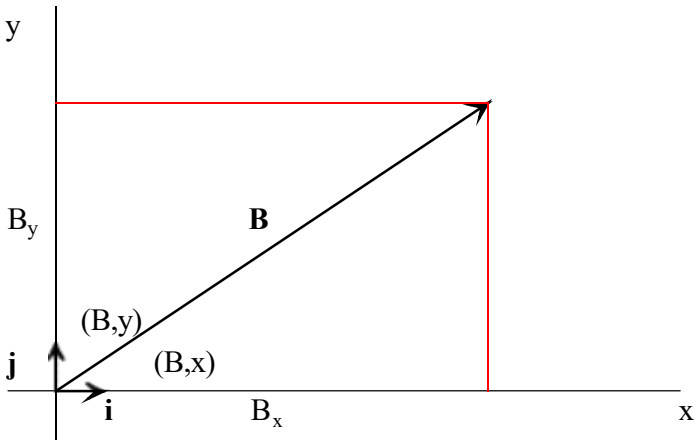


Figure 6.3 Vectors in two-dimensional coordinates

For a general vector $\mathbf{A}$ in three dimensions,It

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad (6.8)$$

$$A_x = A\cos(A,x)$$

$$A_y = A\cos(A,y)$$

$$A_z = A\cos(A,z)$$

$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are vectors with unit lengths in the x, y, and z directions, respectively. A_x, A_y, and A_z are magnitudes in the x, y, and z directions, respectively.

It can be written as above: $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called unit vectors or fundamental vectors. Also, from equation (6.5),

$$A\mathbf{s} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \qquad (6.9)$$

$\mathbf{s}$ is the unit vector in the $\mathbf{A}$ direction.

It can also be written as above. Since $\mathbf{A}$ can be considered as a combination of three vectors $A_x\mathbf{i}$, $A_y\mathbf{j}$, and $A_z\mathbf{k}$ according to the parallelogram rule, the magnitude of vector $\mathbf{A}$ is

$$|\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad (6.10)$$

That is the case.

6.6 Position Vector

A general vector can be expressed as shown in equation (6.8), but a vector $\mathbf{r}$, which is defined by taking the origin of the coordinate system as the starting point of the vector,

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} \qquad (6.11)$$

Considering this, we can see that the coordinates of the endpoint are (r_x, r_y, r_z). Using this, it is possible to express the position of an object as $\mathbf{r}$ instead of the coordinates (x,y,z) that indicate the object's position. Such a vector $\mathbf{r}$ is called a position vector.

6.7 Analytical Addition, Subtraction, and Product with Scalars of Vectors

Vector addition and subtraction are performed using equation (6.8).

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) + (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \qquad (6.12)\end{aligned}$$

$$\begin{aligned}\mathbf{A} - \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) - (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k} \qquad (6.13)\end{aligned}$$

This is how it works. In this way, calculations like **A** + **B** + **C** + **D**... are no matter how much It can also be continued.

The product of a scalar and a vector is,

$$n\mathbf{A} = n(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) = nA_x\mathbf{i} + nA_y\mathbf{j} + nA_z\mathbf{k} \qquad (6.14)$$

This is the result.

6.8 Inner Product of Vectors

The product of the magnitudes of two vectors $\mathbf{A}$ and $\mathbf{B}$ and the cosine of the angle between them,

$$AB\cos(A,B)$$

This is called the inner product or scalar product of $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A} \cdot \mathbf{B}$$

This is how it is written. Using this notation,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos(A,B) \qquad (6.15)$$

It can be written as above. The value of the dot product $\mathbf{A} \cdot \mathbf{B}$ is a scalar. Its geometric meaning is the product of the magnitude of one vector and the component of the other vector in the direction of the former. For example, in Figure 6.4,

$$\mathbf{A} \cdot \mathbf{B} = OP \times OQ$$

That is the case.

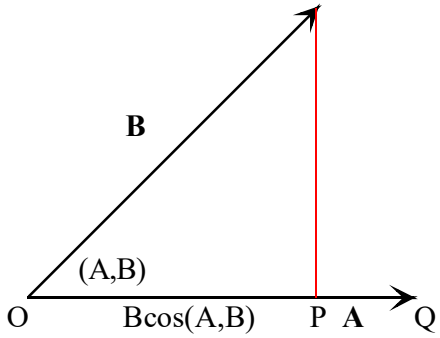


Figure 6.4 Inner product of vectors

The inner product of vectors is:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(Law of commutation)} \tag{6.16}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(Law of distributiveness)} \tag{6.17}$$

This holds true. Also,

$$\mathbf{A} \cdot \mathbf{A} = AA\cos 0 = A^2 \tag{6.18}$$

The holds true. $\mathbf{A} \cdot \mathbf{A}$ is sometimes written as $\mathbf{A}^2$. If $\mathbf{A}$,$\mathbf{B}$ is vertical then $\cos\frac{\pi}{2}=0$,

$$\mathbf{A} \cdot \mathbf{B} = 0 \tag{6.19}$$

This is the result. For a basic vector, the size is 1,

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \tag{6.20}$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 \tag{6.21}$$

This holds true.

The analytical representation of a general vector, equation (6.8), is $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$. In general, the dot product $\mathbf{A} \cdot \mathbf{u}$ of any vector $\mathbf{A}$ and a unit vector $\mathbf{u}$ in any direction is the u-direction component of vector $\mathbf{A}$. Therefore, the x-component A_x of vector $\mathbf{A}$ is:

$$A_x = \mathbf{A} \cdot \mathbf{i} \qquad (6.22)$$

It can be written as. Using this notation, equation (6.8) is,

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{i})\mathbf{i} + (\mathbf{A} \cdot \mathbf{j})\mathbf{j} + (\mathbf{A} \cdot \mathbf{k})\mathbf{k} \qquad (6.23)$$

It can also be written as.

The dot product of $\mathbf{A}$ and $\mathbf{B}$ is,

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= A_xB_x\mathbf{i} \cdot \mathbf{i} + A_xB_y\mathbf{i} \cdot \mathbf{j} + A_xB_z\mathbf{i} \cdot \mathbf{k} \\ &\quad + A_yB_x\mathbf{j} \cdot \mathbf{i} + A_yB_y\mathbf{j} \cdot \mathbf{j} + A_yB_z\mathbf{j} \cdot \mathbf{k} \\ &\quad + A_zB_x\mathbf{k} \cdot \mathbf{i} + A_zB_y\mathbf{k} \cdot \mathbf{j} + A_zB_z\mathbf{k} \cdot \mathbf{k}\end{aligned}$$

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \qquad (6.24)$$

Therefore, using the form of equation (6.22),

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{A} \cdot \mathbf{i})(\mathbf{B} \cdot \mathbf{i}) + (\mathbf{A} \cdot \mathbf{j})(\mathbf{B} \cdot \mathbf{j}) + (\mathbf{A} \cdot \mathbf{k})(\mathbf{B} \cdot \mathbf{k}) \qquad (6.25)$$

It can also be written as.

6.9 Cross Product of Vectors

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, if their magnitudes are equal to the area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$ as two sides, their direction is perpendicular to the planes of this parallelogram, and the vector taken in the direction in which a right-hand screw advances when turned from $\mathbf{A}$ to $\mathbf{B}$ is called the cross product or vector product of $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A} \times \mathbf{B} \qquad (6.26)$$

There are two ways to rotate for $\mathbf{A}$ to $\mathbf{B}$, but we will use the smaller angle of rotation. The cross product of the vectors is $\mathbf{A} \times \mathbf{B}$. This quantity is a vector.

From the definition, The size of $\mathbf{A} \times \mathbf{B}$ is,

$$|\mathbf{A} \times \mathbf{B}| = AB \sin(A,B) \qquad (6.27)$$

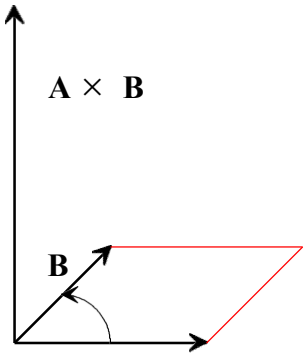


Figure 6.5 Cross product of vectors

The cross product of vectors is:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad (6.28)$$

$$\mathbf{A} \times (\mathbf{B}+\mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad (6.29)$$

$$(\mathbf{A}+\mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \quad (6.30)$$

This holds true. Also,

$$\mathbf{A} \times \mathbf{B} = 0 \quad \text{(When } \mathbf{A} \text{ and } \mathbf{B} \text{ are parallel)} \quad (6.31)$$

$$\mathbf{A} \times \mathbf{A} = 0 \quad (6.32)$$

$$|\mathbf{A} \times \mathbf{B}| = AB \quad \text{(when } \mathbf{A} \text{ and } \mathbf{B} \text{ are perpendicular)} \quad (6.33)$$

This also holds true. For the basic vector,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \quad (6.34)$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j} \quad (6.35)$$

Therefore,

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \quad (6.36)\\ &= A_xB_x\mathbf{i} \times \mathbf{i} + A_xB_y\mathbf{i} \times \mathbf{j} + A_xB_z\mathbf{i} \times \mathbf{k}\\ &\quad + A_yB_x\mathbf{j} \times \mathbf{i} + A_yB_y\mathbf{j} \times \mathbf{j} + A_yB_z\mathbf{j} \times \mathbf{k}\\ &\quad + A_zB_x\mathbf{k} \times \mathbf{i} + A_zB_y\mathbf{k} \times \mathbf{j} + A_zB_z\mathbf{k} \times \mathbf{k}\\ \mathbf{A} \times \mathbf{B} &= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}\end{aligned}$$

The cross product of two vectors is the area of the parallelogram formed by those two vectors, with the direction of the normal vector assigned to it. Generally, we can consider a vector whose magnitude corresponds to the area of a plane figure, and which assigns the direction of the normal vector to that plane. Such a vector is called an area vector. The cross product of two vectors can be considered a type of area vector.

6.10 Product of Three Vectors

Let's consider the product of three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. In such calculations, it's always necessary to remember that the dot product of vectors is a scalar, while the cross product is a vector. For example, $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ can be considered because it's the product of vector $\mathbf{A}$ and scalar $\mathbf{B}\cdot\mathbf{C}$. However, $\mathbf{A}(\mathbf{B}\times\mathbf{C})$ would be the product of vector $\mathbf{A}$ and vector $\mathbf{B}\times\mathbf{C}$, but since the product of two vectors is undefined, it cannot be considered.

The product of three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is let $\mathbf{A}(A_x, A_y, A_z)$, $\mathbf{B}(B_x, B_y, B_z)$ and $\mathbf{C}(C_x, C_y, C_z)$, and try calculating according to the definition. The results of such calculations are listed below.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{6.37}$$

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{A}\times\mathbf{A})\cdot\mathbf{B} = 0 \tag{6.38}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{A}) = -\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0 \tag{6.39}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{6.40}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} \tag{6.41}$$

6.11 Quotient of Vectors

By applying the calculation of the product of three vectors, we can consider the quotient of a vector. In general, this quotient is not uniquely determined, but this is not meaningless. For example, when integrating a function in general, the result is not uniquely determined, but no one would say that integration is meaningless.

(1) Quotient of the dot product

$$\mathbf{A}\cdot\mathbf{X}=\mathrm{k}\quad(\mathbf{A}+0) \tag{6.42}$$

Vector $\mathbf{X}$ that satisfies the following is

$$\mathbf{X}=\frac{k\boldsymbol{A}}{\boldsymbol{A}^2}+\mathbf{C}\times\mathbf{A} \tag{6.43}$$

$\mathbf{C}$ is any vector

[Proof]

$$\mathbf{A}\cdot(\frac{k\boldsymbol{A}}{\boldsymbol{A}^2}+\mathbf{C}\times\mathbf{A})=\mathrm{k}$$
$$\frac{k}{\boldsymbol{A}^2}\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot(\mathbf{C}\times\mathbf{A})=\mathrm{k}$$
$$\mathbf{A}\cdot\mathbf{A}=\mathbf{A}^2,\mathbf{A}\cdot(\mathbf{C}\times\mathbf{A})=0$$
$$\frac{k}{\boldsymbol{A}^2}\mathbf{A}^2+0=\mathrm{k}$$

Therefore, $\mathbf{X}=\frac{k\boldsymbol{A}}{A^2}+\boldsymbol{C}\times\boldsymbol{A}$

(2) Cross product quotient

$$\mathbf{A}\times\mathbf{X}=\mathbf{B}\quad(\mathbf{A}+0) \tag{6.44}$$

A necessary and sufficient condition for a vector **X** that satisfies the given equation is that **A** · **B** = 0, and **A** and **B** are orthogonal. In this case, **X** is

$$\mathbf{X}=\frac{\mathbf{B}\times\mathbf{A}}{\mathrm{A}^2}+\mathrm{c}\mathbf{A} \tag{6.45}$$

c is an arbitrary constant.

That is the case.

[Proof]

$$\mathbf{A}\times\left(\frac{\mathbf{B}\times\mathbf{A}}{\mathrm{A}^2}+\mathrm{c}\mathbf{A}\right)=\mathbf{B}$$

$$\frac{\mathbf{A}\times(\mathbf{B}\times\mathbf{A})}{\mathrm{A}^2}+\mathrm{c}\mathbf{A}\times\mathbf{A}=\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{A})=(\mathbf{A}\cdot\mathbf{A})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{A},$$

and since the condition $\mathbf{A}\cdot\mathbf{B}=0$,

$\mathbf{A}\times(\mathbf{B}\times\mathbf{A})=\mathbf{A}^2\mathbf{B}$, and also $\mathbf{A}\times\mathbf{A}=0$,

$$\frac{A^2B}{A^2}+0=\boldsymbol{B}$$

Therefore, $\mathbf{X}=\frac{\mathbf{B}\times\mathbf{A}}{\mathrm{A}^2}+\mathrm{c}\mathbf{A}$ 。

This result will be used later when we consider magnetic fields.